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THE WIENER-HOPF INTEGRAL EQUATION ON A FINITE INTERVAL: ASYMPTOTIC SOLUTION FOR LARGE INTERVALS WITH AN APPLICATION TO ACOUSTICS

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ABSTRACT

The two-dimensional problem of sound propagation of a monochromatic point source located in air above a flat absorbing earth surface crossed by a rectilinear road with a reflecting cover is considered. The problem is reduced to a Wiener-Hopf integral equation on a finite interval. The behavior of the solution is inves-

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tigated when the the length of the interval tends to infinity. The main term of the asymptotic solution and an estimate for the remainder are obtained.

1. Introduction

The problem of sound propagation in air above the earth's surface crossed by a road is typical in diffraction theory. It can be reduced to a Wiener-Hopf integral equation on a finite interval [1]–[2], where the length of the interval is defined by the width of the road. In [3]–[5] theorems on uniqueness of the solution were proved for the case when the width of the road is significantly less than the length of the acoustic wave. Such a theorem for arbitrary width was proved in [6], where the limiting absorption principle was also justified.

The present paper is devoted to the construction and justification of an asymptotic formula when the length of the interval (the width of the road) tends to infinity. The main particularity of our consideration is that the symbol of the corresponding Wiener-Hopf equation has zeroes of order $\frac{1}{2}$. Note that such problems were considered in [7]–[8] where the complete formal asymptotic expansion was obtained. In contrast to [7]–[8] we not only construct here the main term of the asymptotic expansion, but also give a rigorous estimate for the remainder.

The paper is organized as follows. Section 2 is devoted to a mathematical formulation of the problem under consideration and reduction to a so called modified Wiener-Hopf equation. In section 3 we prove the invertibility of the operator corresponding to the modified Wiener-Hopf equation and obtain an estimate for the norm of the inverse operator in weighted L_2 spaces. Here we use the concept of semisectoriality [9]–[12]. These estimates allow us to formulate and to prove in section 4 the asymptotic representation for the solution of the modified Wiener-Hopf equation. In section 5 we obtain the acoustic field asymptotic representation.

We are very pleased to thank A. Böttcher for extremely useful discussions.

2. Reduction to a modified Wiener-Hopf equation

Let the interval $(0, a)$ on the X axis represent the transversal section of a road on the earth's surface, the air above it lying in the half plane $(x, z) \in R \times R_+$, $R_+ = (0, \infty)$. Let (x_0, z_0), $z_0 > 0$ be the coordinates of a point sound source, $p(x, z)$ be the complex amplitude of the sound pressure (or simply the sound field); let also $k = \omega/c$ be the wave number, $\omega = 2\pi f$ the angular frequency, f the frequency of the source in Hertz, and c the (constant) sound velocity in air.

The problem under consideration is described by the Helmholtz equation in the domain $R \times R_+$,

$$\Delta p(x,z) + k^2 p(x,z) = -\delta(x - x_0)\delta(z - z_0) \tag{2.1}$$

with boundary conditions

$$\frac{\partial p}{\partial z}(x,0) = 0, x \in (0,a), \tag{2.2}$$

$$\frac{\partial p}{\partial z}(x,0) + ikvp(x,0) = 0, x \in R\backslash(0,a), \tag{2.3}$$

where $\mathrm{Re}\, v > 0$, and $v(\in \mathbb{C})$ is the impedance of the earth's surface [13].

To obtain a unique solution we use the so-called *limiting absorption principle* (LAP). For this purpose we add to the wave number in (2.1) a small purely imaginary quantity: $k = k_0 n$, $n = 1 + i\varepsilon$, $k_0 > 0$, $\varepsilon > 0$, simulating the sound decay in air, and look for a solution decreasing at infinity. If such a solution exists, is unique, and has a limit when $\varepsilon \to 0$, we shall say that the LAP is fulfilled for the given problem, and consider this limit as a solution at $\varepsilon = 0$. Below we shall formulate the LAP more precisely.

We restrict ourselves to the two dimensional problem in order to avoid superfluous technical difficulties. This problem corresponds to the acoustic model with a line source.

The solution $p(x,z)$ can be represented in the form

$$p(x,z) = p_\delta(x,z) + \varphi(x,z),$$

where $p_\delta(x,z)$ is the solution of the problem (2.1)-(2.2) whith (2.2) holding for all $x \in R$ (i.e., the road is absent). It is well known [13] that

$$p_\delta(x,z) = \frac{k}{2\pi}\int_R \Phi_\delta(\mu,z) e^{-ik\mu x} d\mu,$$

where

$$\Phi_\delta(\mu,z) = -\frac{e^{ikx_0\mu}}{2\sqrt{2\pi}k\gamma(\mu)}\left[e^{ik|z-z_0|\gamma(\mu)} - V(\mu)e^{ik(z+z_0)\gamma(\mu)}\right], \tag{2.4}$$

$V(\mu) = \frac{v-\gamma(\mu)}{v+\gamma(\mu)}$, $\gamma(\mu) = \sqrt{n^2-\mu^2}$. The branch of the square root $\gamma(\mu)$ is chosen to satisfy the condition $\mathrm{Im}\gamma(\mu) > 0$ when $\varepsilon > 0$, which corresponds to the decreasing of $\Phi_\delta(\mu,z)$ as $z \to \infty$ (the LAP). If $\varepsilon = 0$, then the branch of $\gamma(\mu)$ is defined by continuity.

The unknown function $\varphi(x,z)$ satisfies the following problem:

$$\Delta\varphi(x,z) + k^2\varphi(x,z) = 0, \tag{2.5}$$

$$\frac{\partial \varphi}{\partial z}(x,0) = ikvp_\delta(x,0), \quad x \in [0,a], \tag{2.6}$$

$$\frac{\partial \varphi}{\partial z}(x,0) + ikv\varphi(x,0) = 0, \quad x \in R\backslash[0,a]. \tag{2.7}$$

Let us introduce the dimensionless direct and inverse Fourier transforms:

$$(Ff)(\mu) = \frac{1}{\sqrt{2\pi}} \int_R f(x) e^{ik\mu x} dx, \mu \in R, \tag{2.8}$$

$$(F^{-1}g)(x) = \frac{k}{\sqrt{2\pi}} \int_R g(\mu) e^{-ik\mu x} d\mu, x \in R. \tag{2.9}$$

Let us also consider the weighted function space $L_2(R, \rho)$ with the norm

$$\|f\|_{L_2(R,\rho)} = \left(\int_R |f(\mu)|^2 \rho(\mu) d\mu \right)^{1/2}, \tag{2.10}$$

where $\rho(\mu)$ is the power weight

$$\rho(\mu) = |\mu + i|^{\beta_0} \prod_{j=1}^{m} |\mu - \mu_k|^{\beta_k}, -1 < \beta_k < 1, -1 < \beta_0 + \sum_{k=1}^{m} \beta_k < 1. \tag{2.11}$$

We shall say that the function $f(x, z)$ belongs to the class $C_1 L_2(R \times R_+, \rho)$ if the following conditions hold:

a) $f(x, z)$ has continuous partial derivatives of second order in the region $R \times R_+$;

b) for all $z > 0$, $F\varphi(\cdot, z) \in L_2(R, \rho)$ and there exists $\Phi(\cdot)$ such that $\lim_{z \to 0} \|F\varphi(\cdot, z) - \Phi(\cdot)\|_{L_2(R,\rho)} = 0$;

c) for all $z > 0$, $F\frac{\partial \varphi}{\partial z}(\cdot, z) \in L_2(R, \rho)$ and there exists $\tilde{\Phi}(\cdot)$ such that $\lim_{z \to 0} \left\| F\frac{\partial \varphi}{\partial z}(\cdot, z) - \tilde{\Phi}(\cdot) \right\|_{L_2(R,\rho)} = 0$.

Further we denote $\Phi(\mu) = (F\varphi(\cdot, 0))(\mu)$ and $\tilde{\Phi}(\mu) = (\frac{\partial}{\partial z} F(\cdot, 0))(\mu)$.

The LAP for the given problem is justified in [6] for the space $L_2(R)$. Here we reformulate it for our case.

Theorem 2.1. *Let $k = k_0(1 + i\varepsilon), k_0 > 0, \varepsilon > 0$. Then the solution of the problem (2.5)–(2.7) exists and is unique in the class $C_1 L_2(R \times R_+, \rho)$.*

Theorem 2.2. *If we designate the solution of the problem (2.5)–(2.7) with $\varepsilon > 0$ by $\varphi_\varepsilon(x, z)$, then for any point $(x, z) \in R \times R_+$ there exists a function $\varphi_0(x, z)$ such that*

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(x, z) = \varphi_0(x, z)$$

(where we call the function $\varphi_0(x, z)$ the solution of the problem for $\varepsilon = 0$).

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Henceforth we shall consider $\varepsilon = 0$ unless otherwise stipulated. Let us also write:

$$\begin{aligned} e^{iL\mu}\Phi^+(\mu) : &= (F\chi_{[a,\infty]}\varphi(\cdot,0))(\mu), \\ \Phi^-(\mu) : &= (F\chi_{[-\infty,0]}\varphi(\cdot,0))(\mu), \\ \Phi_L^+(\mu) : &= (F\chi_{[0,a]}\varphi(\cdot,0))(\mu), \end{aligned}$$

where $\chi_I(x)$ is the characteristic function of the interval I and $L = k_0 a$ is a dimensionless quantity.

Applying the Fourier transform (2.8) with respect to the variable x in (2.5)–(2.7) and designating $\Phi(\mu, z) = (F\varphi(\cdot, z))(\mu)$ we obtain

$$\frac{\partial^2 \Phi}{\partial z^2}(\mu, z) + k_0^2(n^2 - \mu^2)\Phi(\mu, z) = 0, \tag{2.12}$$

$$\frac{\partial \Phi}{\partial z}(\mu, 0) = -ik_0 v(e^{iL\mu}\Phi^+(\mu) + \Phi^-(\mu)) + \frac{ik_0 v}{\sqrt{2\pi}}\int_0^a p_\delta(x,0)e^{ik_0\mu x}dx. \tag{2.13}$$

The usual solution of the equation (2.12), satisfying the LAP, is given by $\Phi(\mu, z) = c(\mu)e^{ik_0\gamma(z)\mu}$, so

$$\frac{\partial \Phi}{\partial z}(\mu, 0) = ik_0\gamma(\mu)\Phi(\mu, 0) = ik_0\gamma(\mu)(e^{iL\mu}\Phi^+(\mu) + \Phi_L^+(\mu) + \Phi^-(\mu)). \tag{2.14}$$

Using (2.13), we finally obtain the modified Wiener-Hopf equation:

$$e^{iL\mu}\Phi^+(\mu) + G(\mu)\Phi_L^+(\mu) + \Phi^-(\mu) = (1 - G(\mu))\frac{1}{\sqrt{2\pi}}\int_0^a p_\delta(x,0)e^{ik_0\mu x}dx, \tag{2.15}$$

where $G(\mu) = \frac{\gamma(\mu)}{v + \gamma(\mu)}$. After some transformations,

$$e^{iL\mu}\widetilde{\Phi}^+(\mu) + G(\mu)\widetilde{\Phi}_L^+(\mu) + \widetilde{\Phi}^-(\mu) = f(\mu), \tag{2.16}$$

where

$$\begin{aligned} f(\mu) = \Phi_\delta(\mu, 0) &= -\frac{e^{ik_0(x_0\mu + z_0\gamma(\mu))}}{k_0\sqrt{2\pi}(v + \gamma(\mu))}, \\ e^{iL\mu}\widetilde{\Phi}^+(\mu) &= e^{iL\mu}\Phi^+(\mu) + \frac{1}{\sqrt{2\pi}}\int_a^\infty p_\delta(x,0)e^{ik_0\mu x}dx, \\ \widetilde{\Phi}_L^+(\mu) &= \Phi_L^+(\mu) + \frac{1}{\sqrt{2\pi}}\int_0^a p_\delta(x,0)e^{ik_0\mu x}dx, \\ \widetilde{\Phi}^-(\mu) &= \Phi^-(\mu) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 p_\delta(x,0)e^{ik_0\mu x}dx. \end{aligned}$$

Assume equation (2.16) is satisfied. Taking into account that

$$c(\mu) = \Phi(\mu, 0) = e^{iL\mu}\Phi^+(\mu) + \Phi_L^+(\mu) + \Phi^-(\mu) = (1 - G(\mu))\widetilde{\Phi}_L^+(\mu)$$

and

$$\varphi(x, z) = \frac{k_0}{2\pi}\int\limits_R c(\mu)e^{ik(\gamma(\mu)z - \mu x))d\mu},$$

we finally obtain

$$\varphi(x, z) = \frac{k_0}{2\pi}\int\limits_R \widetilde{\Phi}_L^+(\mu)(1 - G(\mu))e^{-ik_0(\mu x - \gamma(\mu))z}d\mu. \qquad (2.17)$$

3. Estimate for the norm of the inverse operator

Let us consider the following operators: $P^+ := F\chi_{[0,\infty)}F^{-1}$, $P^- := F\chi_{(-\infty,0]}F^{-1}$, $P_{[a,\infty)} := F\chi_{[a,\infty)}F^{-1}$, $P_{[0,a]} := F\chi_{[0,a]}F^{-1}$, $P_L := P_{[0,a]}$, $Q_L := P_{[a,\infty)}$, $P_L^{\perp} := I - P_L$, $T_L(G) := P_L G P_L$, $D_L := P_L^{\perp} + GP_L$. Let us also consider the space $E_{L,\rho} = P_L L_2(R, \rho)$ with the norm induced by the space $L_2(R, \rho)$.

In terms of the operators P_L and $P_L^{\perp}$ the equation (2.16) can be written as

$$D_L\widetilde{\Phi}(\mu) = f(\mu) \qquad (3.1)$$

where $\widetilde{\Phi}_L^+(\mu) = (P_L\widetilde{\Phi})(\mu)$, $e^{iL\mu}\widetilde{\Phi}^+(\mu) = ((Q_L\widetilde{\Phi})(\mu)$, $\widetilde{\Phi}^-(\mu) = (P^-\widetilde{\Phi})(\mu)$. Applying the operator P_L to (3.1) and taking into account the equalities $P_L P_L^{\perp} = P_L^{\perp} P_L = 0$, we have:

$$T_L(G)\widetilde{\Phi}_L^+(\mu) = P_L f(\mu). \qquad (3.2)$$

It is easy to see that the operator Q_L can be written in the form

$$Q_L = e^{iLt}P^+e^{-iLt}. \qquad (3.3)$$

Let us consider $P_L, Q_L, T_L(G)$ as operators acting on the space $L_2(R, \rho)$. It is well known that in this case these operators are bounded. In fact, the singular integral operator S is bounded on the space $L_2(R, \rho)$ [14]–[15]. Since $T_L(G) = P_L G P_L$, $P_L = P^+ - Q_L$, $Q_L = e^{iLt}P^+e^{-iLt}$ by (3.3), $P^+ = \frac{1}{2}(S + I)$. Therefore $P_L, Q_L, T_L(G)$ are bounded in $L_2(R, \rho)$.

Introduce the space of all essentially bounded functions $f(\mu)$ on the real line with the usual norm $\|f(\mu)\|_{L_\infty} = \operatorname{ess\,sup}_{\mu \in R} |f(\mu)|$.

We say a complex function $G(\mu)$ belonging to $L_\infty(R)$ is sectorial if the closure of the set of its essential values in the complex plane $\mathbb{C}$ lies strictly inside a sector with vertex at the origin and angle less than π. It is obvious that if $G(\mu)$ is

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sectorial, then there exist numbers $\sigma \in \mathbb{C}, \delta < 1$, called the sectorial characteristics of the function $G(\mu)$, such that

$$\|\sigma G(\mu) - 1\|_{L_\infty(R)} = \delta. \tag{3.4}$$

Note that the function $G(\mu)$ in (2.16) can be replaced by $\sigma G(\mu)$, where σ is the sectorial characteristic from (3.4). For this we have to replace the unknown function $\widetilde{\Phi}_L^+(\mu)$ by $\sigma^{-1}\widetilde{\Phi}_L^+(\mu)$.

The following theorem is based on the known Brown and Halmos theorem on sectorial operators. The proof given in [12] for the space $L_2(R)$ can be applied to the space $L_2(R,\rho)$ without modifications.

Theorem 3.3. *Let the function $G(\mu) \in L_\infty(R)$ be sectorial with sectorial characteristics σ and δ. Then the operator D_L is invertible in $L_2(R,\rho)$, and the norm of the inverse operator has the following uniform estimate on $L \in (0,\infty)$:*

$$\|D_L^{-1}\|_{L_2(R,\rho)} \leq \sigma(1-\delta)^{-1}. \tag{3.5}$$

The sectorial condition on the function $G(\mu)$ can be weakened.

Theorem 3.4. *Let the functions $h_+(\mu)$ and $h_-(\mu)$ be analytic and bounded in the upper an lower half-planes, respectively, and let the function*

$$\widetilde{G}(\mu) = G(\mu) + e^{iL\mu}h_+(\mu) + e^{-iL\mu}h_-(\mu) \tag{3.6}$$

be sectorial with characteristics $\widetilde{\sigma}$ and $\widetilde{\delta}$. Then the operator D_L is invertible and the estimate (3.5) holds with $\sigma = \widetilde{\sigma}$ and $\delta = \widetilde{\delta}$.

In fact, substituting $G(\mu) = \widetilde{G}(\mu) - e^{iL\mu}h_+(\mu) - e^{-iL\mu}h_-(\mu)$ in (2.16) we obtain an equation of the form (2.16):

$$e^{iL\mu}\widehat{\Phi}^+(\mu) + \widetilde{G}(\mu)\widetilde{\Phi}_L^+(\mu) + \widehat{\Phi}^-(\mu) = f(\mu)$$

with the new $\widehat{\Phi}^+(\mu) = \widetilde{\Phi}^+(\mu) + h_+(\mu)\widetilde{\Phi}_L^+(\mu)$, $\widehat{\Phi}^-(\mu) = \widetilde{\Phi}^-(\mu) + h_-(\mu)e^{-iL\mu}$ $\widetilde{\Phi}_L^+(\mu)$, and sectorial symbol $\widetilde{G}(\mu)$.

Using Theorem 3.4. we shall prove the following important result.

Theorem 3.5. *The operator $T_L(G) : E_{L,\rho} \to E_{L,\rho}$ is invertible, and for L large enough, we have the estimate*

$$\|T_L^{-1}\|_{E_{L,\rho}} \leq \text{const}L^{1/2}. \tag{3.7}$$

Proof. Consider the equation (3.1) where

$$\widetilde{\Phi}(\mu) = e^{iL\mu}\widetilde{\Phi}^{+}(\mu) + \widetilde{\Phi}_L^{+}(\mu) + \widetilde{\Phi}^{-}(\mu) \tag{3.8}$$

(suppose that σ^{-1} is already included in $\Phi_L^{+}(\mu)$). Note that $G(\mu) = \frac{\sqrt{1-\mu^2}}{v+\sqrt{1-\mu^2}}$ is not sectorial. However, it is possible to select functions $h_+(\mu)$, $h_-(\mu)$, analytic and bounded in the upper, and the lower half-planes, respectively, so that the function $\widetilde{G}(\mu) = G(\mu) + h(\mu)$, where $h(\mu) = h_+(\mu)e^{iL\mu} + h_-(\mu)e^{-iL\mu}$, will be sectorial. In particular, it is not difficult to show that if $h_+(\mu) = \frac{2(\cos L + 2\sin L)}{\mu+i} - \frac{5(\cos L+\sin L)}{\mu+2i}$, $h_-(\mu) = -\overline{h_+(\mu)}$ then the function $\widetilde{G}(\mu)$ will be sectorial. Note that in this case $h(\pm 1) = 2i$, $\widetilde{G}(\mu) = G(\mu) + 2i\,(\mathrm{Im}h_+(\mu)\cdot\cos L\mu + \mathrm{Re}h_+(\mu)\cdot \sin L\mu$.

It follows that the operator D_L is invertible according to Theorem 3.4., and $\widetilde{\Phi}(\mu) = (D_L^{-1}f)(\mu)$. Applying the operator P_L to the last equality we obtain $\Phi_L^{+}(\mu) = (P_L D_L^{-1} f)(\mu)$. But $\Phi_L^{+}(\mu) = T_L^{-1}f(\mu)$, so

$$\left\|T_L^{-1}\right\|_{E_{L,\rho}} \leq \left\|D_L^{-1}\right\|_{L_2(R,\rho)} . \tag{3.9}$$

We now obtain an estimate for the norm of D_L^{-1}. To this end we draw a straight line Γ_L with the equation $y = -pL^{1/2}x$. Select $p > 0$ so that $\widetilde{G}(\mu)$ is sectorial relative to Γ_L and the inequality

$$\rho(I\widetilde{G}(\mu), \Gamma_L) \geq mL^{1/2} \tag{3.10}$$

is satisfied for some constant m independent of L, where $I\widetilde{G}(\mu)$ is the image of the function $\widetilde{G}(\mu)$, and ρ is the distance from a set to a line. It is enough to consider the graph of the image of the function $\widetilde{G}(\mu)$ in a small neighborhood of the points $\mu = \pm 1$. In these neighborhoods $\widetilde{G}(\mu) = \frac{\sqrt{2}}{v}\sqrt{1\mp\mu} + 2i\cos l(1\mp\mu) + \underline{\underline{O}}(|1\mp\mu|)$. Let us make the change of variables $u = 1 \mp \mu$. Then in a neighborhood of $u = 0$ the image of $\widetilde{G}(\mu)$ may be parameterized as

$$\begin{cases} x := \mathrm{Im}\widetilde{G}(\mu) = 2\cos Lu + \mathrm{Im}\frac{\sqrt{2}}{\nu}\sqrt{u} + \underline{\underline{O}}(u), \\ y := \mathrm{Re}\widetilde{G}(\mu) = \mathrm{Re}\frac{\sqrt{2}}{\nu}\sqrt{u} + \underline{\underline{O}}(u). \end{cases} \tag{3.11}$$

We have eight cases: $u = 1 \pm \mu$, $\mathrm{sign}\,\mathrm{Im}\,\nu = \pm 1$, $\mathrm{sign}\,u = \pm 1$. Consider first $u = 1 - \mu$, $\mathrm{Im}\,\nu > 0$, $u > 0$. Drawing the curve corresponding to (3.11), we see that it is enough to show (3.10) for $u \in [0, \frac{2\pi}{L}]$. The distance $\rho(I\widetilde{G}(\mu), \Gamma_L)$ is calculated by the formula

$$\rho(I\widetilde{G}(\mu), \Gamma_L) = \frac{2\cos Lu + \mathrm{Im}\frac{\sqrt{2}}{\nu}\sqrt{u} + pL^{1/2}\mathrm{Re}\frac{\sqrt{2}}{\nu}\sqrt{u} + \underline{\underline{O}}(u)}{\sqrt{1+p^2L}}.$$

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We choose p so that the main part of the numerator increases monotonically in the interval $u \in [0, \frac{2\pi}{L}]$ (for instance, set $p = \frac{4}{\mathrm{Re}\frac{1}{\nu}}$). The minimum of the main part of the expression is attained in this case at $u = 0$, and the estimate (3.10) is true. The remaining seven cases are considered similarly.

From Theorem 3.4. we have

$$\|D_L^{-1}\|_{L_2(R,\rho)\to L_2(R,\rho)} \le \frac{1}{\rho(I\widetilde{G}(\mu), \Gamma_L)} \le \mathrm{const} \cdot L^{1/2}.$$

This estimate and the inequality (3.9) complete the proof of the theorem. □

4. Asymptotics for the solution of the modified Wiener-Hopf equation

Introduce the operators $T(G) = P^+GP^+$, $J : J(f(x)) = f(-x)$, $W_L = e^{iLx}JP_L$. Note that the operator $T(G)$ is formally the limit of the operator $T_L(G)$ when $L \to \infty$. However, if $\varepsilon = 0$, the operators $T(G)$ and $T_L(G)$ act on the different spaces (we shall explain this below in more detail). This fact is one of the basic difficulties involved.

Introduce weighted spaces $L_{2,s_1,s_2} := L_2(R, \rho_{s_1,s_2})$ and $L_{2,s} := L_{2,s,s}$ with weight

$$\rho_{s_1,s_2}(\mu) = |1-\mu|^{s_1}\,|1+\mu|^{s_2}\,|\mu+i|^{-(s_1+s_2)}, \tag{4.1}$$

$s_1, s_2 \in (-1,1)$, of the kind (2.11). Also introduce the corresponding spaces $E_{L,s_1,s_2} := P_L L_{2,s_1,s_2}$, $E_{L,s} := P_L L_{2,s}$.

We say that the continuous, bounded function $G(\mu)$, defined on the real axis, has a standard canonical factorization if it can be represented as

$$G(\mu) = G_+(\mu) \cdot G_-(\mu), \tag{4.2}$$

where $G_+(\mu), G_+^{-1}(\mu)$ and $G_-(\mu), G_-^{-1}(\mu)$ are analytic in the upper and lower half-planes, respectively, and bounded in the corresponding closed half-planes.

It is well known [14]–[15] that in this case the operator $T(G)$ is invertible in $L_{2,s}^+ := P^+(L_{2,s})$, $s \in (-1,1)$, and for the inverse operator we have the formula

$$T^{-1}(G) = \frac{1}{G_+}P^+\frac{1}{G_-}P^+. \tag{4.3}$$

In our case the function $G(\mu)$ has also a Wiener-Hopf factorization, but the presence of zeroes of half-integer order leads to the fact that the operator $T(G)$ acts from one weighted space to another. More precisely, the following result holds.

Theorem 4.6. *The function $G(\mu)$ of type (2.15) admits the factorization*

$$G(\mu) = G_+(\mu)\cdot G_-(\mu). \tag{4.4}$$

Here $G_\pm(\mu) = \frac{\sqrt{1\pm\mu}}{a_\pm(\mu)}$ where the functions $a_+^{\pm1}(\mu)$ and $a_-^{\pm1}(\mu)$ are analytic in the upper Π_+ and lower Π_- half-planes, respectively, and continuous in $\overline{\Pi}_+$ and $\overline{\Pi}_-$, except at the point $\mu=\infty$, where the following conditions hold:

$$|a_\pm(\mu)| = O(|\mu|^{1/2}), |a_\pm^{-1}(\mu)| = O(|\mu|^{-1/2}).$$

The operator $T(G): L^+_{2,s+1}\to L^+_{2,s}$, $s\in(-1,0)$, is a bounded invertible operator whose inverse $T^{-1}(G): L^+_{2,s}\to L^+_{2,s+1}$ has the form (4.3).

Proof. Represent the function G as $G(\mu) = \frac{\sqrt{1-\mu^2}}{\sqrt{1+\mu^2}}G_0(\mu)$, where $G_0(\mu) = \frac{\sqrt{1+\mu^2}}{v+\sqrt{1-\mu^2}}$. It is easy to see that the function $G_0(\mu)$ is Hölder continuous (with Hölder exponent 1/2) on the closed real line $\dot{R}$, $G_0(\mu)\neq 0$, $\mu\in R$, and

$$\operatorname{ind} G_0(\mu) := \frac{1}{2\pi}\arg G_0(\mu)|_R = 0.$$

By a classical result of F. D. Gakhov (see [14]–[15]) $G_0(\mu)$ admits a canonical factorization (4.2),

$$G_0(\mu) = G_{0,+}(\mu)\cdot G_{0,-}(\mu).$$

Writing $a_\pm(\mu) = \frac{G^{-1}_{0,\pm}(\mu)}{\sqrt{1\mp i\mu}}$ we obtain (4.4).

The last statement of the theorem can be obtained from (4.4) by standard techniques ([14]–[15]). Theorem 4.6. is proved. □

We write also $K(G) = T^{-1}(G) - T(G^{-1})$, considering this as an operator from $L^+_{2,s}$ into $L^+_{2,s+1}$, where $s\in(-1,0)$.

Define the operator

$$B_L(G) := P_L T^{-1}(G) P_L + W_L K(G) W_L. \tag{4.5}$$

It is well known [14] that if $G(\mu)$ is continuous on the closed real axis $\dot{R}$ and $T^{-1}(G)$ exists, then the equality

$$T_L(G)B_L(G) = P_L - E_L(G), \tag{4.6}$$

is true, where $E_L(G) = P_L T(G) Q_L K(G) P_L + W_L T(G) Q_L K(G) W_L$.

Theorem 4.7. *Let $G(\mu)$ and $f(\mu)$ be as in equation (2.16). Then the equation (3.2) has a unique solution in the space $E_{L,s}$, $s\in(0,1)$, and the following estimate,*

$$\left\|\Phi_L^+ - B_L f\right\|_{L_{2,s}} \le M L^{-s/2}, s\in(0,1), \tag{4.7}$$

holds, where $M = M(s)$ is independent of $L\ge L_0>0$.

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We now prove some len

Lemma 4.1. *Let $b(t)\in L_2$*
tation holds:

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Proof. By definition,

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and $Q_L = e^{iLt}P^+e^{-iL\tau}$. T

we obtain formula (4.11). T

Assume $\Phi_0(\mu) = B_L f$. Then it is not difficult to verify that

$$T_L(\Phi_L^+ - \Phi_0) = E_L f. \tag{4.8}$$

Taking into account that if $s_1 < s_2$ then $L_{2,s_1} \subset L_{2,s_2}$ and that the inclusion operator is bounded, we shall consider the operator E_L acting from $E_{L,s}$ to $E_{L,s+1}$, $s \in (-1,0)$, the operators T_L and T_L^{-1} acting from $E_{L,s+1}$ to $E_{L,s+1}$ and the function $\Phi_L^+ \in L_{2,s+1}$, $s \in (-1,0)$. From (4.8) we have $\Phi_L^+ - \Phi_0 = T_L^{-1} E_L f$.

Using (4.3), it is not difficult to show that the right part of the equation (4.8) is represented as follows:

$$E_L f = E_L^{(1)} f + E_L^{(2)} f,$$

where $E_L^{(1)} = P_L G Q_L \frac{1}{G_+} P^- \frac{1}{G_-} P_L$, $E_L^{(2)} = W_L G Q_L \frac{1}{G_+} P^- \frac{1}{G_-} W_L$. Then

$$\Phi_L^+ - \Phi_0 = X_L^{(1)} + X_L^{(2)}, \tag{4.9}$$

where $X_L^{(1)}$, $X_L^{(2)}$ are solutions to the problems

$$T_L X_L^{(1,2)} = E_L^{(1,2)} f. \tag{4.10}$$

Introduce the notation $f_L = P_L f$, $b_L^- = P^- G_-^{-1} f_L$, $d_L = b_L^- a_+^{-1}$, $\varphi_L = Q_L G^{-1} \sqrt{1-\tau}\, d_L$. Then $E_L^{(1)} f = P_L G \varphi_L$.

We now prove some lemmas.

Lemma 4.1. *Let $b(t) \in L_{2,s}$, $s \in (-1,1)$. Then the following integral representation holds:*

$$(Q_L b)(t) = \frac{1}{2\pi i} \int_R \frac{b(t) - b(\tau)}{t - \tau} e^{iL(t-\tau)} d\tau. \tag{4.11}$$

Proof. By definition,

$$(P^+ b)(t) = \frac{1}{2\pi i} \int_R \frac{b(\tau)}{\tau - t} d\tau + \frac{1}{2} b(t)$$

and $Q_L = e^{iLt} P^+ e^{-iL\tau}$. Taking into account the equality

$$-\frac{1}{i\pi} \int_R \frac{e^{iL(t-\tau)}}{\tau - t} d\tau = 1$$

we obtain formula (4.11). The proof is finished. □

Lemma 4.2. *We have the following integral representation:*

$$b_L^-(-1-i\xi) = \frac{1}{2\pi i}\int\limits_R f_L(\tau)K(\tau,\xi)d\tau, \qquad (4.12)$$

where $\xi \geq 0$, $K(\tau,\xi) = \frac{G_-^{-1}(\tau)-G_-^{-1}(-1-i\xi)}{\tau+1+i\xi}$.

Proof. Since $b_L^- = P^-\frac{1}{G_-}f_L$, for $\xi > 0$ we have

$$b_L^-(-1-i\xi) = -\frac{1}{2\pi i}\int\limits_R f_L(\tau)\frac{G_-^{-1}(\tau)}{\tau+1+i\xi}d\tau.$$

But if $\xi > 0$ then $\int\limits_R \frac{f_L(\tau)}{\tau+1+i\xi}d\tau = 0$, whence under the condition $\xi > 0$ we obtain (4.12). At $\xi = 0$ and $\tau = \pm 1$, the kernel $K(\tau,\xi)$ has a weak singularity, so the integral on the right side of (4.12) is continuous at $\xi = 0$. The proof is finished. □

Lemma 4.3. *The following estimate holds:*

$$\left|b_L^-(-1)\right| \leq \text{const}\left(\|f_L\|_C + \|f_L\|_{L_2(R)}\right). \qquad (4.13)$$

Proof. According to the previous lemma,

$$b_L^-(-1) = \frac{1}{2\pi i}\int\limits_R f_L(\tau)K(\tau,0)d\tau.$$

We split the last integral into two integrals on the sets $\tau \in [-2,2]$ and $\tau \in R\backslash[-2,2]$ and apply the Hölder inequality to the second integral:

$$|b_L^-(-1)| \leq \tfrac{1}{2\pi}$$
$$\left(\|f_L\|_C \int_{-2}^{2}|K(\tau,0)|d\tau + \|f_L\|_{L_2(R)}\left(\int\limits_{R\backslash(-2,2)}|K(\tau,0)|^2d\tau\right)^{1/2}\right)$$

This gives (4.13).

Lemma 4.4. *The following estimate holds:*

$$|d_L(-1-i\xi) - d_L(-1)| \leq \text{const}\left(\|f_L\|_C + \|f_L\|_{L_2}\right)\left|\sqrt{\xi}\right|, \xi \in [0,1]. \quad (4.14)$$

Proof. It is not difficult
$a_1(-1-i\xi)\sqrt{\xi}$, where
this it follows that for ξ

$d_L(-1-i\xi)-d_L(-1) =$

Because $b_L(-1-i\xi$
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$|b_L(-1-i\xi) - b_L(-1)$

Using the inequaliti

$\left|\frac{G_-^{-1}(-1)}{\tau+}\right.$

$\int_{-2}^{0}\frac{d\tau}{|\sqrt{\tau+1}|\,|\sqrt{\tau-1}|}$

we arrive at the estimate
from (4.13), (4.15) and
(4.14). The proof is fini

Lemma 4.5. *If the fun*
$\|f_L\|_{L_2(R)}$, $\|f_L\|_C$ *are*

Proof. It is obvious that

f_L

where $R_L(\tau,t) = \frac{e^{-iL}}{}$

$\|f_L\|_C \leq \text{const}\left(\left|f(\right.\right.$

$+\|f\|_{L_2(R)}\left(\int\limits_{|\tau-t|>1}\right.$

$\leq \text{const}\left(\|f\|_{L_2(R)}\right.$

Proof. It is not difficult to show that if $\xi \in [0,1]$, then $a_-^{-1}(-1-i\xi) = a_0 + a_1(-1-i\xi)\sqrt{\xi}$, where $a_1(-1-i\xi)$ is bounded for $\xi \in [0,1]$, $a_1(-1) \neq 0$. From this it follows that for $\xi \in [0,1]$,

$$d_L(-1-i\xi)-d_L(-1) = a_0(b_L(-1-i\xi)-b_L(-1))+\sqrt{\xi}a_1(-1-i\xi)b_L(-1-i\xi). \tag{4.15}$$

Because $b_L(-1-i\xi) - b_L(-1) = \frac{1}{2\pi i}\int\limits_R f_L(\tau)\,(K(\tau,\xi) - K(\tau,0))d\tau$, carrying out simple calculations, we obtain

$$|b_L(-1-i\xi) - b_L(-1)| \leq \frac{|\xi|}{2\pi}\int\limits_R |f_L(\tau)| \left|\frac{G_-^{-1}(-1) - G_-^{-1}(\tau)}{\tau+1}\right| \frac{1}{|\tau+1+i\xi|}d\tau.$$

Using the inequalities

$$\left|\frac{G_-^{-1}(-1) - G_-^{-1}(\tau)}{\tau+1}\right| \leq \frac{\text{const}}{|\sqrt{\tau+1}|\,|\sqrt{\tau-1}|}, \quad \tau \in R,$$

$$\int\limits_{-2}^{0} \frac{d\tau}{|\sqrt{\tau+1}|\,|\sqrt{\tau-1}|\,|\tau+1+i\xi|} \leq \int\limits_{-1}^{1} \frac{d\tau}{|\sqrt{\tau}|\,|\tau+i\xi|} \leq \frac{\text{const}}{|\sqrt{\xi}|}, \quad \xi \in (0,1],$$

we arrive at the estimate $\left|b_L^-(-1-i\xi) - b_L^-(-1)\right| \leq \text{const}\,\|f_L\|_C\,|\sqrt{\xi}|$, and then from (4.13), (4.15) and the boundedness of $a_1(-1-i\xi)$ at $\xi \in [0,1]$ we obtain (4.14). The proof is finished. □

Lemma 4.5. *If the function $f(\mu)$ is defined as in formula (2.16), then the norms $\|f_L\|_{L_2(R)}$, $\|f_L\|_C$ are uniformly bounded for $z_0 > 0$, $x_0 \in R$.*

Proof. It is obvious that $\|f_L\|_{L_2(R)} \leq \|f\|_{L_2(R)}$. Using the integral representation

$$f_L = P_L f = -\frac{e^{iLt}}{2\pi i}\int_R R_L(\tau,t)f(\tau)d\tau,$$

where $R_L(\tau,t) = \frac{e^{-iL\tau} - e^{-iLt}}{\tau - t}$, we obtain

$$\|f_L\|_C \leq \text{const}\left(\left|f(t)\int\limits_{|\tau-t|\leq 1} R_L(\tau,t)d\tau + \int\limits_{|\tau-t|\leq 1} R_L(\tau,t)\,(f(\tau)-f(t))\,d\tau\right| + \|f\|_{L_2(R)}\left(\int\limits_{|\tau-t|>1} |R_L(\tau,t)|^2\,d\tau\right)^{1/2}\right) \leq$$

$$\leq \text{const}\left(\|f\|_{L_2(R)} + \|f\|_C + \sup_{|\tau-t|\leq 1}\left|\frac{f(\tau)-f(t)}{\sqrt{\tau-t}}\right|\right)$$

It is not difficult to show that if the function $f(\mu)$ is defined as in (2.16), then all terms on the right side of the last inequality are uniformly bounded for $z_0 > 0$. The proof is finished. □

Let us turn to the proof of theorem 4.7.. The following estimates are true:

$$\left\|X_L^{(1)}\right\|_{L_{2,s}} \le \text{const} \cdot L^{-s/2}, \left\|X_L^{(2)}\right\|_{L_{2,s}} \le \text{const} \cdot L^{-s/2}, s \in (0,1). \quad (4.16)$$

First we prove an estimate for $X_L^{(1)}$. Using the integral representation (4.11) for Q_L we have

$$\varphi_L(t) = (Q_L \frac{1}{G}\sqrt{1-\tau}d_L)(t) = \frac{1}{2\pi i}\int_R \frac{\frac{v-\sqrt{1-t^2}}{\sqrt{1-t^2}} - \frac{v-\sqrt{1-\tau^2}}{\sqrt{1-\tau^2}}}{t-\tau}$$

$$\sqrt{1-\tau}d_L(\tau)e^{iL(t-\tau)}d\tau.$$

Deforming the path of integration R into $\Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are the rays $-1-i\xi, \xi \in (-\infty, 0)$, traversed in opposite directions, we obtain

$$\varphi_L(t) = c_0 e^{iL(t+1)} \int_0^\infty d_L(-1-i\xi)\frac{e^{-L\xi}}{\sqrt{\xi}(t+1+i\xi)}d\xi, \quad (4.17)$$

where $c_0 = -\frac{v}{\pi i\sqrt{i}}$. We divide the last integral in two parts,

$$I_{L,1}(t) = d_L(-1)\int_0^\infty \frac{e^{-L\xi}}{\sqrt{\xi}(t+1+i\xi)}d\xi,$$

$$I_{L,2}(t) = \int_0^\infty \frac{d_L(-1-i\xi) - d_L(-1)}{\sqrt{\xi}} \cdot \frac{e^{-L\xi}d\xi}{t+1+i\xi},$$

and introduce the notation $\varphi_{L,1}(t) = c_0e^{iL(t+1)}I_{L,1}(t), \varphi_{L,2}(t) = c_0e^{iL(t+1)}I_{L,2}(t)$, $E_{L,1}^{(1)} = P_LG\varphi_{L,1}, E_{L,2}^{(1)} = P_LG\varphi_{L,2}$. As before $X_{L,1}^{(1)}$ and $X_{L,2}^{(1)}$ let be solutions with right parts $E_{L,1}^{(1)}$ and $E_{L,2}^{(1)}$ respectively. Using the technique found in [16], p. 525-526, we derive the integral formula

$$\int_0^\infty \frac{e^{-Lu}}{\sqrt{u}(\varepsilon-u)}du = \frac{\pi}{\sqrt{-\varepsilon}}e^{-\varepsilon L}\left(\Phi\left(\sqrt{-\varepsilon L}\right) - 1\right). \quad (4.18)$$

Therefore

$$\int_0^\infty \frac{e^{-L\xi}}{\sqrt{\xi}(t+1+i\xi)}d\xi = -\frac{\pi\sqrt{-i}}{\sqrt{t+1}}e^{-iL(t+1)}\left(\Phi\left(\sqrt{-iL(t+1)}\right) - 1\right), \quad (4.19)$$

where $\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}$

$\varphi_{L,1}(t) = -c_0\pi\sqrt{-}$

According to [17],

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so we have

$\varphi_{L,1}(t) = c_0\sqrt{\pi}d_L(-1$

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Let us estimate the L

$\left\|P_LG(\varphi_{L,1} - \Delta X_{L,1}^{(1)}\right.$

Split the last integral on

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$\le$ const $\cdot |d_L(-1)|^2\int_{|u}$

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$\le$ const $\cdot |b_L^-(-1)|^2 L$

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$\Delta X_{L,1}^{(1)}(t)$.

It is not hard to obtai

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:fined as in (2.16), then all
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□

ving estimates are true:

$L^{-s/2}, s \in (0,1)$. (4.16)

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where Γ_1, Γ_2 are the rays
is, we obtain

$\frac{e^{-L\xi}}{t+1+i\xi)} d\xi,$ (4.17)

parts,

$d\xi,$

$\cdot \frac{e^{-L\xi} d\xi}{t+1+i\xi},$

), $\varphi_{L,2}(t) = c_0 e^{iL(t+1)} I_{L,2}(t)$,
$_{,1}$ and $X^{(1)}_{L,2}$ let be solutions
technique found in [16], p.

$\overline{-\varepsilon L}) - 1)$. (4.18)

$\overline{-iL(t+1)}) - 1)$, (4.19)

where $\Phi(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^z e^{-t^2} dt$ is the Fresnel integral. Hence

$$\varphi_{L,1}(t) = -c_0 \pi \sqrt{-i} \frac{1}{\sqrt{t+1}} d_L(-1) \left(\Phi \left(\sqrt{-i(t+1)L} \right) - 1 \right). \tag{4.20}$$

According to [17],

$$\Phi(\sqrt{iz}) - 1 = -\frac{e^{-iz}}{\sqrt{\pi i z}} \left(1 + O\left(\frac{1}{z}\right) \right), |z| \gg 1,$$

so we have

$$\varphi_{L,1}(t) = c_0 \sqrt{\pi} d_L(-1) \frac{1}{\sqrt{L}} \frac{e^{iL(t+1)}}{t+1} \left(1 + O\left(\frac{1}{L(t+1)} \right) \right), |L(t+1)| \gg 1. \tag{4.21}$$

Let us consider the function $\Delta X^{(1)}_{L,1}(t) = c_0 \sqrt{\pi} d_L(-1) \cdot \frac{e^{iL(t+1)} - (e^{iL(t+1)} - 1)/(iL(t+1))}{\sqrt{L}(t+1)}$ with the same asymptotics as $\varphi_{L,1}(t)$ at $|L(t+1)| \gg 1$. It is evident that $\Delta X^{(1)}_{L,1} = P_L \Delta X^{(1)}_{L,1}$, so

$$T_L(X^{(1)}_{L,1} - \Delta X^{(1)}_{L,1}) = P_L G(\varphi_{L,1} - \Delta X^{(1)}_{L,1}). \tag{4.22}$$

Let us estimate the $L_{2,s}$ norm, $s \in (0,1)$, of the right part of (4.22):

$$\left\| P_L G(\varphi_{L,1} - \Delta X^{(1)}_{L,1}) \right\|^2_{L_{2,s}} \le \int\limits_R |G(t)|^2 \cdot \left| \varphi_{L,1}(t) - \Delta X^{(1)}_{L,1}(t) \right|^2 \cdot \rho_s(t) dt. \tag{4.23}$$

Split the last integral on the sets $|t+1| < \frac{1}{L}$ and $|t+1| \ge \frac{1}{L}$ and consider the second one in detail:

$$\begin{aligned} &\int_{|t+1| \ge L^{-1}} |G(t)|^2 \cdot \left| \varphi_{L,1}(t) - \Delta X^{(1)}_{L,1}(t) \right|^2 \cdot \rho_s(t) dt \\ &\le \text{const} \cdot |d_L(-1)|^2 \int_{|u| \ge L^{-1}} \left| \sqrt{-i\pi} \cdot \frac{\Phi(\sqrt{iLu}) - 1}{\sqrt{u}} + \frac{e^{iLu}}{\sqrt{L}u} - \frac{e^{iLu} - 1}{iL\sqrt{L}u^2} \right|^2 |u|^{1+s} du \\ &\le \text{const} \cdot \left| b^-_L(-1) \right|^2 \frac{1}{L^{1+s}} \int\limits_{|u| \ge 1} \left| \sqrt{-i\pi} \cdot \frac{\Phi(\sqrt{iu}) - 1}{\sqrt{u}} + \frac{e^{iu}}{u} - \frac{e^{iu} - 1}{iu^2} \right|^2 |u|^{1+s} du \\ &\le \text{const} \cdot \left| b^-_L(-1) \right|^2 \frac{1}{L^{1+s}}, s \in (0,1). \end{aligned}$$

The last integral converges due to the presence of the "compensatory" term $\Delta X^{(1)}_{L,1}(t)$.

It is not hard to obtain an analogous estimate for the integral on the set $|t+1| < \frac{1}{L}$. Therefore,

$$\left\| P_L G(\varphi_{L,1} - \Delta X^{(1)}_{L,1}) \right\|_{L_{2,s}} \le \text{const} L^{-\frac{s+1}{2}} \left| b^-_L(-1) \right|. \tag{4.24}$$

One can verify also that $\left\|\Delta X^{(1)}_{L,1}\right\|_{L_{2,s}} \leq \text{const}L^{-s/2}\left|b^-_L(-1)\right|$. Hence using (4.24) and Theorem 3.4. we have $\left\|X^{(1)}_{L,1}\right\|_{L_{2,s}} \leq \text{const}L^{-s/2}\left|b^-_L(-1)\right|$, $s \in (0,1)$. Finally, by Lemmas 4.3. and 4.5.,

$$\left\|X^{(1)}_{L,1}\right\|_{L_{2,s}} \leq \text{const}L^{-s/2}\left(\|f_L\|_C + \|f_L\|_{L_2}\right) \leq \text{const}L^{-s/2}, s \in (0,1). \tag{4.25}$$

Let us now estimate $\left\|E^{(1)}_{L,2}f\right\|_{L_{2,s}}$. It is enough to consider the integral of $\varphi_{L,2}(t)$ on the interval (0,1), because the integral on $(1,\infty)$ decreases exponentially when $L \to \infty$ uniformly on t. Taking into account Lemma 4.4. and the equality $\|P_L\|_{L_{2,s}} = 1$ we have the following chain of inequalities:

$$\begin{aligned}
\left\|E^{(1)}_{L,2}f\right\|^2_{L_{2,s}} &\leq \int_R |G(u)|^2 \left|\int_0^1 \frac{d_L(-1-i\xi) - d_L(-1)}{\sqrt{\xi}}\frac{e^{-L\xi}}{u+i\xi}d\xi\right|^2 \rho_s(u)du \\
&\leq \text{const}(\|f_L\|_C + \|f_L\|_{L_2})^2 \int_R |G(u)|^2 \left|\int_0^\infty e^{-Lu\eta}d\eta\right|^2 \rho_s(u)du \\
&\leq \frac{\text{const}}{L^2}(\|f_L\|_C + \|f_L\|_{L_2})^2 \cdot \int_R |G(u)|^2 \frac{1}{u^2}\rho_s(u)du \\
&\leq \frac{\text{const}}{L^2}(\|f_L\|_C + \|f_L\|_{L_2})^2, s \in (0,1).
\end{aligned}$$

Using Theorem 3.4., Lemma 4.5. and the fact that $X^{(1)}_{L,2}$ is a solution for the equation of type (4.10) with right part $E^{(1)}_{L,2}f$, we arrive at the estimate

$$\left\|X^{(1)}_{L,2}\right\|_{L_{2,s}} \leq \text{const}L^{-1/2}(\|f_L\|_C + \|f_L\|_{L_2(R)}) \leq \text{const}L^{-s/2}, s \in (0,1). \tag{4.26}$$

Since $X^{(1)}_L = X^{(1)}_{L,1} + X^{(1)}_{L,2}$, from (4.25), (4.26) and Lemma 4.5. we now have

$$\left\|X^{(1)}_L\right\|_{L_{2,s}} \leq \text{const}\,L^{-s/2}(\|f_L\|_C + \|f_L\|_{L_2(R)}) \leq \text{const}L^{-s/2}, s \in (0,1). \tag{4.27}$$

Similarly we obtain an estimate for $\left\|X^{(2)}_L\right\|_{L_{2,s}}$. Finally, the equality $\Phi^+_L - \Phi_0 = X^{(1)}_L + X^{(2)}_L$ yields

$$\left\|\Phi^+_L - B_L f\right\|_{L_{2,s}} \leq \text{const}L^{-s/2}, s \in (0,1). \tag{4.28}$$

Theorem 4.7. is thus completely proved. □

5. Asymptotics fo

Substituting the expansi
obtain

$$\varphi(x,z) = \frac{k_0}{2\pi}\int_R (B_I$$

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From Theorem 4.7. we r

Theorem 5.8. *Let* $p(x,$
the LAP. Then

$$p(x,z) = \frac{k_0}{2\pi}\int_R$$
$$+ \frac{k_0}{2\pi}\int_R$$

where $\Phi_\delta(\mu,z)$ *is define*
and the remainder $O(x,$

where the constant $M =$

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5. Asymptotics for the solution of the original problem

Substituting the expansion $\Phi_L^+ = B_L f + (\Phi_L^+ - B_L f)$ into formula (2.17), we obtain

$$\varphi(x,z) = \frac{k_0}{2\pi}\int\limits_R (B_L f)(\mu)(1-G(\mu))e^{-ik_0(\mu x - \gamma(\mu)z)}d\mu + O(x,z), \quad (5.1)$$

where the first integral represents the principal term for the asymptotics of the perturbed field, and the residual term $O(x,z)$ is estimated as follows:

$$|O(x,z)| = \frac{k_0}{2\pi}\left(\int\limits_R \left|\Phi_L^+ - B_L f\right|^2 \rho_s(\mu)d\mu\right)^{1/2} \times$$

$$\left(\int\limits_R |1-G(\mu)|^2 \left|e^{-2ik_0(\mu x - \gamma(\mu)z)}\right| \rho_s^{-1}(\mu)d\mu\right)^{1/2}$$

From Theorem 4.7. we now obtain our main result.

Theorem 5.8. *Let $p(x,z)$ be the solution to the problem (2.1)-(2.3) which satisfies the LAP. Then*

$$\begin{aligned} p(x,z) &= \frac{k_0}{2\pi}\int\limits_R \Phi_\delta(\mu,z)e^{-ik_0\mu x}d\mu \\ &+ \frac{k_0}{2\pi}\int\limits_R (B_L f)(\mu)(1-G(\mu))e^{-ik_0(\mu x-\gamma(\mu)z)}d\mu + O(x,z), \end{aligned}$$

where $\Phi_\delta(\mu,z)$ is defined by (2.4), the operator $B_L = B_L(G)$ has the form (4.5) and the remainder $O(x,z)$ admits the estimate

$$|O(x,z)| \le ML^{-s/2}, s \in (0,1),$$

where the constant $M = M(s)$ is independent of $x, x_0 \in R$ and of $z, z_0 > 0$.

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A NOTE C

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The purpose of this note
problem and give some (

Let $\mathcal{H}$ be a complex sepa
space of $\mathcal{H}$ valued square
χ denote the character or
H^2 given by

A subspace (always supp
$S\mathcal{M} \subset \mathcal{M}$. An S-invaria
J is a multiplication oper

with J a measurable fu
Since **J** maps H^2 into it
([2]).

We shall assume in the r
are all isometries in wh
operator inner functions
that $\cap_{n=1}^{\infty} \mathbf{J}^n H^2 = \{0\}$.
[5]). The isometry S is