Estimates for the condition numbers of large semi-definite Toeplitz matrices

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This paper is devoted to asymptotic estimates for the condition numbers

$$\kappa\Big(T_n(a)\Big) = \|T_n(a)\| \, \|T_n^{-1}(a)\|$$

of large $n \times n$ Toeplitz matrices $T_n(a)$ in the case where $a \in L^\infty$ and $\operatorname{Re} a \geq 0$. We describe several classes of symbols a for which $\kappa(T_n(a))$ increases like $(\log n)^\alpha$, n^α, or even $e^{\alpha n}$. The consequences of the results for singular values, eigenvalues, and the finite section method are discussed. We also consider Wiener-Hopf integral operators and multidimensional Toeplitz operators.

1. Introduction.

A bounded operator A on a Hilbert space H is called *positive semi-definite* if the real part of (Af, f) is nonnegative for every $f \in H$, i.e., if $\operatorname{Re}(Af, f) \geq 0$ for all $f \in H$, and it is said to be *positive definite* if there is an $\varepsilon > 0$ such that $\operatorname{Re}(Af, f) \geq \varepsilon(f, f)$ for all $f \in H$.

We here consider the case where A is given by a Toeplitz matrix on $H = l^2$. Let $\mathbf{T}$ be the complex unit circle and denote by $L^\infty := L^\infty(\mathbf{T})$ the essentially bounded functions on $\mathbf{T}$. For $a \in L^\infty$, let $\{a_n\}_{n \in \mathbf{Z}}$ stand for the sequence of the Fourier coefficients,

$$a_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta}) e^{-in\theta} d\theta,$$

and define the *finite Toeplitz matrices* $T_n(a)$ ($n \in \mathbf{N} := \{1, 2, 3, \ldots\}$) and the *infinite Toeplitz matrix* $T(a)$ by

$$T_n(a) := (a_{j-k})_{j,k=0}^{n-1}, \quad T(a) := (a_{j-k})_{j,k=0}^{\infty}.$$

We tacitly identify the matrices $T_n(a)$ and $T(a)$ with the operators they induce on $\mathbf{C}^n = l^2(\{0, 1, \ldots, n-1\})$ and $l^2 := l^2(\{0, 1, 2, \ldots\})$. The boundedness of $T(a)$ follows from the boundedness of a. The function a is usually referred to as the *symbol* of the matrices/operators $T_n(a)$ and $T(a)$.

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We equip the space $L^2 := L^2(\mathbf{T})$ with the scalar product and the norm

$$(f,g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta, \quad \|f\|_2 := (f,f)^{1/2}.$$

Let $\mathcal{P}_n$ and H^2 denote the functions in L^2 whose sequence of Fourier coefficients is supported in $\{0,1,\ldots,n-1\}$ and $\{0,1,2,\ldots\}$, respectively. Clearly, we can also think of $T_n(a)$ and $T(a)$ as acting on $\mathcal{P}_n$ and H^2, respectively. Denoting by $P_n : L^2 \to \mathcal{P}_n$ and $P : L^2 \to H^2$ the orthogonal projections, we can write

$$T_n(a)f = P_n(af) \quad (f \in \mathcal{P}_n), \quad T(a)f = P(af) \quad (f \in H^2),$$

which implies that

$$\Big(T_n(a)f, f\Big) = \int_{-\pi}^{\pi} a(e^{i\theta})|f(e^{i\theta})|^2\,d\theta \quad (f \in \mathcal{P}_n), \tag{1}$$

$$\Big(T(a)f, f\Big) = \int_{-\pi}^{\pi} a(e^{i\theta})|f(e^{i\theta})|^2\,d\theta \quad (f \in H^2). \tag{2}$$

From these two equalities we deduce that $T_n(a)$ and $T(a)$ are positive semi-definite if only $\operatorname{Re} a \ge 0$ a.e. and that these operators are positive definite whenever $\operatorname{Re} a \ge \varepsilon$ a.e. for some $\varepsilon > 0$.

Given, $a \in L^\infty$, let $\mathcal{R}(a)$ be the essential range of a, i.e., the spectrum of a as an element of the Banach algebra L^∞. We denote by $\operatorname{conv}\mathcal{R}(a)$ the convex hull of $\mathcal{R}(a)$. Obviously, $\operatorname{conv}\mathcal{R}(a)$ is always a compact and convex set. Put

$$\operatorname{dist}\Big(0, \operatorname{conv}\mathcal{R}(a)\Big) = \min\Big\{|\lambda| : \lambda \in \operatorname{conv}\mathcal{R}(a)\Big\}.$$

Let $\partial\operatorname{conv}\mathcal{R}(a)$ be the boundary of $\operatorname{conv}\mathcal{R}(a)$. It is readily seen that

$$\operatorname{dist}\Big(0, \operatorname{conv}\mathcal{R}(a)\Big) > 0 \Longleftrightarrow \exists\gamma \in \mathbf{T}\ \ \exists\varepsilon > 0 : \operatorname{Re}(\gamma a) \ge \varepsilon \ \text{ a.e.}, \tag{3}$$

$$0 \in \partial\operatorname{conv}\mathcal{R}(a) \ \text{ or } \ \operatorname{dist}\Big(0, \operatorname{conv}\mathcal{R}(a)\Big) \ge 0 \Longleftrightarrow \exists\gamma \in \mathbf{T} : \operatorname{Re}(\gamma a) \ge 0 \ \text{ a.e..} \tag{4}$$

The following simple fact is well known.

Proposition 1.1. *Suppose $a \in L^\infty$ does not vanish identically and* $\operatorname{conv}\mathcal{R}(a)$ *is not a line segment containing the origin in its interior. If* $\operatorname{dist}(0, \operatorname{conv}\mathcal{R}(a)) \ge 0$ *or* $0 \in \partial\operatorname{conv}\mathcal{R}(a)$, *then $T_n(a)$ is invertible for every $n \ge 1$.*

Proof. Assume $T_n(a)$ is not invertible. Then there is a nonzero $f \in \mathcal{P}_n$ such that $T_n(a)f = 0$, and (1) gives $\int a|f|^2 = 0$. By virtue of (4) there exists a $\gamma \in \mathbf{T}$ such that $\operatorname{Re}(\gamma a) \ge 0$ a.e. Hence $\operatorname{Re}(\gamma a)|f|^2 = 0$ a.e., and as f vanishes almost nowhere, we deduce that $\operatorname{Re}(\gamma a) = 0$ a.e. Consequently, $\operatorname{conv}\mathcal{R}(\gamma a) = i[m, M]$ with real numbers m and M. Because $\int \operatorname{Im}(\gamma a)|f|^2 = 0$ and f vanishes almost nowhere, it follows that $m < 0$ and $M > 0$, which means that $\mathcal{R}(a)$ is a line segment containing the origin in its interior. However, this case was excluded. ■

We remark that if $\operatorname{conv} \mathcal{R}(a)$ is a line segment containing the origin in its interior then it may happen that $T_n(a)$ is singular for infinitely many n. More about this will be said in the Appendix.

The *condition number* $\kappa(A)$ of a bounded Hilbert space operator A is defined by

$$\kappa(A) := \|A\| \, \|A^{-1}\|;$$

we put $\kappa(A) = \|A^{-1}\| = \infty$ in case A is not invertible. Proposition 1.1 describes a class of symbols a for which $\kappa(T_n(a))$ is finite for every $n \geq 1$. The next result is also well known.

Proposition 1.2. *If $a \in L^\infty$ and*

$$\limsup_{n \to \infty} \kappa\Big(T_n(a)\Big) < \infty \tag{5}$$

then $T(a)$ is invertible.

Proof. Suppose a does not vanish identically. The operators $T_n(a)P_n$ converge strongly to $T(a)$ on H^2 as $n \to \infty$. Therefore

$$0 < \|a\|_\infty = \|T(a)\| \leq \liminf_{n \to \infty} \|T_n(a)\| \leq \limsup_{n \to \infty} \|T_n(a)\| \leq \|a\|_\infty$$

and hence $\|T_n(a)\| \to \|a\|_\infty > 0$. Thus

$$\limsup_{n \to \infty} \kappa\Big(T_n(a)\Big) < \infty \iff \limsup_{n \to \infty} \|T_n^{-1}(a)\| < \infty.$$

Now assume there is an $M < \infty$ such that $\|T_n^{-1}(a)\| \leq M$ for all $n \geq n_0$. Then for every $f \in H^2$,

$$\|P_n f\| \leq \|T_n^{-1}(a)\| \, \|T_n(a)P_n f\| \leq M\|T_n(a)P_n f\|,$$

and passing to the strong limit, we get $\|f\| \leq M\|T(a)f\|$. Considering adjoints we obtain analogously that $\|f\| \leq M\|T^*(a)f\|$. This proves that $T(a)$ is invertible. ■

Thus, if the condition numbers $\kappa(T_n(a))$ remain bounded as $n \to \infty$, then $T(a)$ must necessarily be invertible. Criteria for $T(a)$ to be invertible are known for large classes of symbols a (see, e.g., [5] and [4]). We only remark that $T(a)$ is never invertible if a has a zero, i.e., if $0 \in \mathcal{R}(a)$ (*Hartman-Wintner theorem*).

As the following result reveals, things are very simple for positive definite Toeplitz operators (and their rotations).

Proposition 1.3 (Brown-Halmos theorem). *Let $a \in L^\infty$ and suppose*

$$d := \operatorname{dist}\Big(0, \operatorname{conv} \mathcal{R}(a)\Big) > 0. \tag{6}$$

Then $T(a)$ is invertible and

$$\kappa\Big(T(a)\Big) \leq \frac{\|a\|_\infty}{d}\left(1 + \sqrt{1 - \frac{d^2}{\|a\|_\infty^2}}\,\right) < \frac{2\|a\|_\infty}{d}, \tag{7}$$

the operators $T_n(a)$ are invertible and

$$\kappa\Big(T_n(a)\Big) \le \frac{\|a\|_\infty}{d}\left(1+\sqrt{1-\frac{d^2}{\|a\|_\infty^2}}\,\right) < \frac{2\|a\|_\infty}{d} \ \textit{ for all } \ n \ge 1, \tag{8}$$

and if, in addition, a is piecewise continuous, then

$$\lim_{n\to\infty} \kappa\Big(T_n(a)\Big) = \kappa\Big(T(a)\Big). \tag{9}$$

Proof outline. There is a $\gamma \in \mathbf{T}$ such that the set $\gamma \operatorname{conv} \mathcal{R}(a)$ is contained in the set

$$\Big\{z \in \mathbf{C} : \operatorname{Re} z \ge d, \ \ |z| \le \|a\|_\infty\Big\}.$$

Multiplying the latter set by $\lambda := d/\|a\|_\infty^2$ we obtain a set contained in the disk

$$\{z \in \mathbf{C} : |z-1| < r\}, \quad r := \sqrt{1 - d^2/\|a\|_\infty^2}.$$

Hence

$$\|\lambda\gamma T_n(a) - I\| \le \|\lambda\gamma T(a) - I\| \le \|\lambda\gamma a - 1\|_\infty \le r < 1,$$

which implies the invertibility of $T_n(a)$ and $T(a)$ and shows that the norms of the inverses are at most

$$|\lambda\gamma|\frac{1}{1-r} = \frac{d}{\|a\|_\infty^2}\frac{1+r}{1-r^2} = \frac{1}{d}\left(1+\sqrt{1-\frac{d^2}{\|a\|_\infty^2}}\,\right).$$

Since $\|T_n(a)\| \le \|T(a)\| \le \|a\|_\infty$, this gives (7) and (8).

Gohberg and Feldman (see [5, Theorem II.5.1]) showed that under the hypothesis of the proposition the condition numbers $\kappa(T_n(a))$ remain bounded. That the limit of $\kappa(T_n(a))$ exists and equals $\kappa(T(a))$ was proved in [15] and [2]. ■

In fact, the converse of Proposition 1.2, i.e., the implication

$$T(a) \ \text{ is invertible } \implies \limsup_{n\to\infty} \kappa\Big(T_n(a)\Big) < \infty \tag{10}$$

is true for large classes of symbols a essentially violating (6). For instance, (10) holds if

$$a \in (C + H^\infty) \cup (C + \overline{H^\infty}) \cup PQC$$

or if a is locally sectorial over QC (see [5] and [4] and the references therein). Moreover, the implication

$$T(a) \ \text{ is invertible } \implies \lim_{n\to\infty} \kappa\Big(T_n(a)\Big) = \kappa\Big(T(a)\Big)$$

is also valid in many cases, for example if $a \in PQC$ or if a is locally normal over QC (see [15] and [2]). Notice, however, that all these results pertain to the case where $u = \operatorname{Re} a$ (and thus, all the more, a itself) has no zeros.

Now suppose $a \in L^\infty \setminus \{0\}$ and all we know is that $u(\theta) := \operatorname{Re} a(e^{i\theta}) \geq 0$ for almost all θ. If a has a zero $e^{i\theta_0}$ (and therefore u has the zero θ_0), then $T(a)$ is not invertible due to the Hartman-Wintner theorem, whence

$$\limsup_{n\to\infty} \kappa\Big(T_n(a)\Big) = \infty \tag{11}$$

by virtue of Proposition 1.2.

The case where u has a zero θ_0 but $a(e^{i\theta_0}) \neq 0$ (see Section 3 for the meaning of this inequality) is less transparent. Proceeding as in the proof of Proposition 1.1 and using the F. and M. Riesz theorem which says that nonzero functions in H^2 vanish almost nowhere, we see that $T(a)$ has necessarily a trivial kernel (i.e., is injective) provided $\operatorname{conv}\mathcal{R}(a)$ is not a line segment containing the origin in its interior. However, $T(a)$ may be invertible or may not be invertible.

Example 1.4. Let $a(e^{i\theta}) = \sin|\theta| + i\cos\theta$. In that case $u(\theta) = \sin|\theta|$ has two zeros in $(-\pi, \pi]$ but $|a(e^{i\theta})| = 1$ for all θ. Notice that as θ goes from $-\pi$ to π, the image $a(e^{i\theta})$ traverses the half-circle $\{z \in \mathbf{C} : \operatorname{Re} z \geq 0,\ |z| = 1\}$ first from $-i$ to i and then back from i to $-i$. The operator $T(a)$ is invertible (see, e.g., [5, Theorem I.7.1] or [4, Theorem 2.42]) and one can show that

$$\lim_{n\to\infty} \kappa\Big(T_n(a)\Big) = \kappa\Big(T(a)\Big)$$

(see [15] and [2]). ■

Example 1.5. The symbol $a(e^{i\theta}) = e^{i\theta/2} = \cos(\theta/2) + i\sin(\theta/2)$ traverses the half-circle $\{z \in \mathbf{C} : \operatorname{Re} z \geq 0,\ |z| = 1\}$ from $-i$ to i and then jumps back to $-i$ as θ moves from $-\pi$ to π $(+0)$. Clearly, a itself has no zero, but $u(\theta) = \cos(\theta/2)$ vanishes at $\theta = -\pi = \pi$ (mod 2π). Because the line segment between the endpoints of the jumps of a passes through the origin ("hidden zero"), the operator $T(a)$ is not invertible (see, e.g., [5, Theorem IV.2.1] or [4, Theorem 2.74]). Thus, $\kappa(T_n(a))$ cannot be bounded. We will return to this example in Section 4. ■

In this paper we establish estimates for the growth of $\kappa(T_n(a))$ provided $a \in L^\infty$, $\operatorname{Re} a \geq 0$ a.e., and $u = \operatorname{Re} a$ or both $u = \operatorname{Re} a$ and a have zeros.

2. The idea behind the approach

The idea of our approach is extremely simple. The purpose of this section is to illustrate this idea by an example.

Consider the Toeplitz matrices

$$T_n(a) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

The symbol is

$$a(e^{i\theta}) = -e^{i\theta} + 2 - e^{-i\theta} = 2\,(1 - \cos\theta), \quad \theta \in (-\pi, \pi].$$

Thus, $a \geq 0$ but a has a zero at $\theta = 0$. Since

$$\frac{2}{\pi^2}\,\theta^2 \leq 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \leq \frac{1}{2}\,\theta^2, \tag{12}$$

the "order" of the zero is 2.

The operator $T(a)$ is not invertible and hence (11) holds. The eigenvalues of $T_n(a)$ can be calculated explicitly: they are

$$\lambda_k^{(n)} = 2 - 2\cos\frac{k\pi}{n+1} = 4\sin^2\frac{k\pi}{2(n+1)} \quad (k = 1, \ldots, n)$$

(see [8, Example 5.3]). Consequently,

$$\|T_n^{-1}(a)\| = 1/\lambda_1^{(n)} \sim \frac{1}{\pi^2}n^2,$$

where $x_n \sim y_n$ means that $x_n/y_n \to 1$ as $n \to \infty$. Because $\|T_n(a)\| \to \|T(a)\| = \|a\|_\infty = 4$, it follows that

$$\kappa\Big(T_n(a)\Big) \sim \frac{4}{\pi^2}n^2 \;\text{ as }\; n \to \infty. \tag{13}$$

Here now is how we proceed.

Upper estimate. We replace a by $a + ig_n$ where

$$g_n(e^{i\theta}) = \cos n\theta = (e^{in\theta} + e^{-in\theta})/2.$$

As the Fourier coefficients $(g_n)_k$ of g_n are zero for $|k| \leq n-1$, we have

$$T_n(a) = T_n(a + ig_n).$$

Clearly,

$$d_n := \operatorname{dist}\Big(0, \operatorname{conv}\mathcal{R}(a + ig_n)\Big) > 0.$$

Our aim is to estimate d_n from below.

The graph of $a + ig_n$ in $\mathbf{C} = \mathbf{R}^2$ is given by

$$(2 - 2\cos\theta,\ \cos n\theta), \quad \theta \in (-\pi, \pi]. \tag{14}$$

Put

$$\varepsilon_n := \frac{2}{3}\left(1 - \cos\frac{\pi}{3n}\right) = \frac{4}{3}\sin^2\frac{\pi}{6n}. \tag{15}$$

The graph of

$$\left(2 - 2\cos\theta,\ \frac{1}{2} - \frac{1}{\varepsilon_n}(2 - 2\cos\theta)\right), \quad \theta \in \mathbf{R} \tag{16}$$

is the straight line $y = 1/2 - (1/\varepsilon_n)x$. We show that the range of $a + ig_n$ lies above this line. By virtue of (14) and (16) this is equivalent to showing that

$$\frac{1}{\varepsilon_n}(2 - 2\cos\theta) + \cos n\theta > \frac{1}{2} \tag{17}$$

for $\theta \in (-\pi, \pi]$. If $|n\theta| < \pi/3$, then $\cos n\theta > 1/2$ and hence (17) is true. If $|n\theta| \geq \pi/3$, then $\cos\theta \leq \cos(\pi/(3n))$, whence, by (15),

$$\frac{1}{\varepsilon_n}(2 - 2\cos\theta) + \cos n\theta \geq \frac{1}{\varepsilon_n}\left(2 - 2\cos\frac{\pi}{3n}\right) - 1 = 3 - 1 > \frac{1}{2},$$

which gives (17) again.

Thus, $d_n \geq D_n$ where D_n is the distance of the origin to the sraight line $y = 1/2 - (1/\varepsilon_n)x$. Obviously,

$$D_n = \frac{\varepsilon_n}{4}\frac{1}{\sqrt{1/4 + \varepsilon_n^2/4}},$$

and since $\varepsilon_n \to 0$ as $n \to \infty$, we have $D_n > \varepsilon_n/4$ for all sufficiently large n. Taking into account (8) we therefore obtain

$$\begin{aligned} \kappa\big(T_n(a)\big) &= \kappa\big(T_n(a + ig_n)\big) < \frac{2\|a\|_\infty}{d_n} = \frac{8}{d_n} \leq \frac{8}{D_n} \\ &< \frac{32}{\varepsilon_n} = \frac{24}{\sin^2(\pi/(6n))} < 216\, n^2. \blacksquare \end{aligned} \tag{18}$$

Lower estimate. We can construct an even trigonometric polynomial p_m^3 of degree $3m$ such that

$$\|p_m^3\|_\infty = (m+1)^3, \quad \|p_m^3\|_2^2 \geq \frac{16}{9}m^5, \quad |p_m^3(\theta)| < \frac{8}{\theta^3} \ \text{ for } \ \theta \neq 0.$$

Starting with (12) we get

$$\begin{aligned} 2\pi\|ap_m^3\|_2^2 &= \int_{-\pi}^{\pi} |a(e^{i\theta})|^2 |p_m^3(\theta)|^2 d\theta \leq \frac{1}{2}\int_0^{\pi} \theta^4 |p_m^3(\theta)|^2 d\theta \\ &< \frac{1}{2}\frac{1}{m^4}(m+1)^6 \int_0^{1/m} d\theta + 8^2 \int_{1/m}^{\pi} \theta^4\theta^{-6} d\theta \\ &\leq \frac{1}{2}\frac{1}{m^4}2^6 m^6 \frac{1}{m} + 64m < 128m \\ &\leq 128m\frac{9}{16m^5}\|p_m^3\|_2^2 = \frac{72}{m^4}\|p_m^3\|_2^2. \end{aligned}$$

Given $n \geq 9$, write $n = 3m + k$ with $k \in \{1, 2, 3\}$. We then have

$$\|T_n(a)p_m^3\|_2^2 = \|P_n(ap_m^3)\|_2^2 \leq \|ap_m^3\|_2^2 \leq \frac{36}{\pi}\frac{1}{m^4}\|p_m^3\|_2^2 \leq \frac{36}{\pi}\left(\frac{4}{n}\right)^4 \|p_m^3\|_2^2,$$

whence

$$\|T_n^{-1}(a)\| \geq \frac{\sqrt{\pi}}{6}\left(\frac{n}{4}\right)^2$$

and thus, for sufficiently large n,

$$\kappa\big(T_n(a)\big) \geq \frac{\|a\|_\infty}{\sqrt{\pi}}\frac{\sqrt{\pi}}{6}\left(\frac{n}{4}\right)^2 = \frac{1}{24}n^2. \blacksquare \tag{19}$$

Of course, (18) and (19) are more coarse than (13). However, as will be shown in what follows, the simple arguments we used to derive (18) and (19) work as nicely as above in many situations in which the eigenvalues are not available explicitly.

3. Upper estimates for the condition numbers

Let $a \in L^\infty$ and suppose $\operatorname{Re} a \geq 0$ a.e. Put

$$u(\theta) := \operatorname{Re} a(e^{i\theta}), \quad v(\theta) := \operatorname{Im} a(e^{i\theta}), \quad \theta \in \mathbf{R}.$$

Thus, $u(\theta) \geq 0$ for almost all $\theta \in \mathbf{R}$.

A number $\theta_0 \in (-\pi, \pi]$ is said to be a zero of u if

$$\operatorname{ess\,inf} \Big\{ u(\theta) : |\theta - \theta_0| < \delta \Big\} = 0 \ \text{ for each } \ \delta > 0.$$

Assume u has only finitely many zeros $\theta_1, \ldots, \theta_N$ in $(-\pi, \pi]$. Then

$$\operatorname{ess\,inf} u > 0 \ \text{ on } \ (-\pi, \pi] \setminus \bigcup_{j=1}^{N} (\theta_j - \delta, \theta_j + \delta)$$

for each $\delta > 0$. Fix a $\delta > 0$ so that the sets $(\theta_j - \delta, \theta_j + \delta)$ are pairwise disjoint and define functions $\omega_j : \mathbf{N} \to (0, \infty)$ by

$$\frac{1}{\omega_j(n)} := \operatorname{ess\,inf} \Big\{ u(\theta) : \frac{1}{n} < |\theta - \theta_j| < \delta \Big\}. \tag{20}$$

The function ω_j characterizes the "order" of the zero θ_j. Clearly, $\omega_j(n) \to \infty$ as $n \to \infty$.

In what follows we write $f(\theta) \simeq g(\theta)$ to indicate that there is a constant $K \in [1, \infty)$ independent of θ such that

$$0 < (1/K) g(\theta) \leq f(\theta) \leq K g(\theta).$$

The notation $f(n) \simeq g(n)$ means that there exists a constant $K \in [1, \infty)$ such that

$$0 < (1/K) g(n) \leq f(n) \leq K g(n) \ \text{ for all } \ n \in \mathbf{N}.$$

Example 3.1: powerlike zeros. If $u(\theta) \simeq |\theta - \theta_j|^\alpha$ for $|\theta - \theta_j| < \delta$ with some $\alpha > 0$, then $\omega_j(n) \simeq n^\alpha$. In case

$$u(\theta) \simeq |\theta - \theta_j|^\alpha \ \text{ for } \ \theta \in (\theta_j, \theta_j + \delta), \ \ u(\theta) \simeq |\theta - \theta_j|^\beta \ \text{ for } \ \theta \in (\theta_j - \delta, \theta_j),$$

we have $\omega_j(n) \simeq n^{\max\{\alpha, \beta\}}$. ■

Example 3.2: logarithmic zeros. Let $\alpha > 0$ and

$$u(\theta) \simeq 1 \Big/ |\log(|\theta - \theta_j|/\pi)|^\alpha \ \text{ for } \ 0 < |\theta - \theta_j| < \delta.$$

In that case $\omega_j(n) \simeq (\log n)^\alpha$. ■

Example 3.3: exponential zeros. Suppose $\gamma > 0$, $\alpha > 0$ and

$$u(\theta) \simeq e^{-\gamma/|\theta-\theta_j|^\alpha} \text{ for } 0 < |\theta - \theta_j| < \delta.$$

Then $\omega_j(n) \simeq e^{\gamma n^\alpha}$. ■

Here is the main result of this section.

Theorem 3.4. *Let $a \in L^\infty$, suppose $\operatorname{Re} a \geq 0$ a.e., and assume $u := \operatorname{Re} a$ has exactly $N \geq 1$ zeros $\theta_1, \ldots, \theta_N$ on $(-\pi, \pi]$. Define $\omega_j(n)$ by (20), put $\omega(n) := \max\{\omega_1(n), \ldots, \omega_N(n)\}$, and let $v := \operatorname{Im} a$. Then*

$$\kappa\Big(T_n(a)\Big) \leq 12\|a\|_\infty\Big(\|v\|_\infty + 1\Big)\omega(13^{N+1}n).$$

for all sufficiently large n.

In the case $N = 1$ we can modify the trick we used in Section 2 to establish (18). In the $N > 1$ case we need an additional tool. We don't know whom the following result has to be attributed to. It was presented as a problem to the 1977 International Mathematics Olympiad by the Polish members of the Scientific Committee. The proof, given for the sake of completeness, is from [10, Problem 5.14].

Lemma 3.5. *Let $\beta_1, \ldots, \beta_N$ be real numbers and $\mu > 0$. Then there exists a number $q \in \mathbf{N}$ such that $1 \leq q \leq ([1/\mu] + 1)^N$ and*

$$q\beta_j \in \mathbf{Z} + (-\mu, \mu) \text{ for all } j \in \{1, \ldots, N\}.$$

Proof. For $x \in \mathbf{R}$, denote by $[x]$ and $\{x\}$ the integral and fractional part of x, respectively. Thus, $x = [x] + \{x\}$ with $[x] \in \mathbf{Z}$ and $\{x\} \in [0, 1)$.

Put $K = [1/\mu] + 1$ and divide the cube $[0, 1)^N$ into K^N congruent cubes of the form

$$[i_1/K, (i_1 + 1)/K) \times \ldots \times [i_N/K, (i_N + 1)/K). \tag{21}$$

The $K^N + 1$ points

$$\Big(\{l\beta_1\}, \ldots, \{l\beta_N\}\Big), \quad l = 0, 1, \ldots, K^N,$$

all belong to $[0, 1)^N$ and therefore two of them must be located in the same cube (21). Consequently, there are l_1, l_2 such that $0 \leq l_1 < l_2 \leq K^N$ and

$$-1/K \leq l_2\beta_j - l_1\beta_j < 1/K \text{ for all } j.$$

Put $q := l_2 - l_1$ and $m_j := [l_2\beta_j] - [l_1\beta_j]$. Then

$$|q\beta_j - m_j| = \Big|l_2\beta_j - [l_2\beta_j] - \Big(l_1\beta_j - [l_1\beta_j]\Big)\Big| = |\{l_2\beta_j\} - \{l_1\beta_j\}| < 1/K < \mu. \blacksquare$$

Proof of Theorem 3.4. Using Lemma 3.5 with $\mu = 1/12$ and $\beta_j = n\theta_j/(2\pi)$ we get an integer q_n such that

$$1 \leq q_n \leq 13^N \text{ and } nq_n\theta_j \in 2\pi\mathbf{Z} + \Big(-\frac{\pi}{6}, \frac{\pi}{6}\Big). \tag{22}$$

We have

$$\cos(nq_n\theta) = \cos(nq_n\theta_j)\cos\Big(nq_n(\theta-\theta_j)\Big) - \sin(nq_n\theta_j)\sin\Big(nq_n(\theta-\theta_j)\Big)$$

and (22) shows that

$$\cos(nq_n\theta_j) > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin(nq_n\theta_j) < \sin\frac{\pi}{6} = \frac{1}{2}.$$

If $|nq_n(\theta-\theta_j)| < \pi/6$, then

$$\cos\Big(nq_n(\theta-\theta_j)\Big) > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin\Big(nq_n(\theta-\theta_j)\Big) < \sin\frac{\pi}{6} = \frac{1}{2},$$

and hence we arrive at the conclusion that

$$\cos(nq_n\theta) > \frac{1}{2} \text{ whenever } |\theta-\theta_j| < \frac{\pi}{6nq_n}. \tag{23}$$

Recall that v is the imaginary part of a. Put

$$\frac{1}{\varepsilon_{n,j}} := 3\Big(\|v\|_\infty + 1\Big)\omega_j\Big(\frac{6nq_n}{\pi}\Big), \quad M := 2\Big(\|v\|_\infty + 1\Big) \tag{24}$$

and consider the function

$$b_n(e^{i\theta}) := a(e^{i\theta}) + iM\cos(nq_n\theta). \tag{25}$$

Since $q_n \geq 1$, we have $T_n(a) = T_n(b_n)$. Now let n be so large that $\pi/(6nq_n) < \delta$. We claim that the essential range $\mathcal{R}(b_n|(\theta_j-\delta,\,\theta_j+\delta))$ lies above the straight line given by $y = 1-(1/\varepsilon_{n,j})x$. As

$$b_n(e^{i\theta}) = u(\theta) + i\Big(v(\theta) + M\cos(nq_n\theta)\Big),$$

this is equivalent to saying that

$$v(\theta) + M\cos(nq_n\theta) > 1 - \frac{1}{\varepsilon_{n,j}}u(\theta)$$

for almost all $\theta \in (\theta_j-\delta,\,\theta_j+\delta)$. We prove that actually

$$\frac{1}{\varepsilon_{n,j}}u(\theta) + M\cos(nq_n\theta) > 1 + \|v\|_\infty. \tag{26}$$

for almost all $\theta \in (\theta_j-\delta,\,\theta_j+\delta)$.

If $|\theta-\theta_j| < \pi/(6nq_n)$ then (23), (24), and the nonnegativity of $u(\theta)$ give

$$\frac{1}{\varepsilon_{n,j}}u(\theta) + M\cos(nq_n\theta) > \frac{M}{2} = \|v\|_\infty + 1.$$

So let $\pi/(6nq_n) < |\theta - \theta_j| < \delta$. Then $u(\theta) \geq 1/\omega_j(6nq_n/\pi)$ by (20), whence, by (24),

$$\begin{aligned}\frac{1}{\varepsilon_{n,j}}u(\theta) + M\cos(nq_n\theta) &\geq \frac{1}{\varepsilon_{n,j}\omega_j(6nq_n/\pi)} - M \\ &= 3\Big(\|v\|_\infty + 1\Big) - 2\Big(\|v\|_\infty + 1\Big) = \|v\|_\infty + 1.\end{aligned}$$

This completes the proof of (26).

Thus, the essential range of the restriction of b_n to $\bigcup_j(\theta_j - \delta,\, \theta_j + \delta)$ lies above the line

$$y = 1 - \frac{1}{\varepsilon_n}x \quad \text{where} \quad \varepsilon_n := \min_j \varepsilon_{n,j} \tag{27}$$

(here we took also into account that $\operatorname{Re} b_n \geq 0$). The number η given by

$$\eta := \operatorname{ess\,inf}\Big\{u(\theta) : \theta \in (-\pi, \pi] \setminus \bigcup_{j=1}^{N}(\theta_j - \delta,\, \theta_j + \delta)\Big\}$$

is positive. If $\theta \in (-\pi, \pi] \setminus \bigcup_j(\theta_j - \delta,\, \theta_j + \delta)$, then $b_n(e^{i\theta})$ is located on the right of the vertical line $x = \eta$. Since $1/\varepsilon_n \to \infty$ as $n \to \infty$, it follows that $\mathcal{R}(b_n)$ is contained in the half-plane above the line (27) for all sufficiently large n.

The distance of the origin to the line (27) is $D_n = \varepsilon_n/\sqrt{1 + \varepsilon_n^2}$ and thus $D_n > \varepsilon_n/2$ if only n is large enough. Hence, for all sufficiently large n we obtain from Proposition 1.3 that $T_n(a) = T_n(b_n)$ is invertible and that

$$\begin{aligned}\|T_n^{-1}(a)\| &< \frac{2}{D_n} < \frac{4}{\varepsilon_n} = 12\Big(\|v\|_\infty + 1\Big)\max_j \omega_j\Big(\frac{6nq_n}{\pi}\Big) \\ &= 12\Big(\|v\|_\infty + 1\Big)\omega\Big(\frac{6}{\pi}13^N n\Big) \leq 12\Big(\|v\|_\infty + 1\Big)\omega\Big(13^{N+1}n\Big). \blacksquare\end{aligned}$$

Example 3.6. Let $a(e^{i\theta}) = \sin|\theta| + iv(\theta)$ where $v \in L^\infty$ is any real-valued function. The function $u(\theta) := \sin|\theta|$ has exactly two zeros in $(-\pi, \pi]$, $\theta_1 = 0$ and $\theta_2 = \pi$. Since

$$\frac{1}{\omega_1(n)} = \frac{1}{\omega_2(n)} = \inf\Big\{\sin\theta : \frac{1}{n} < \theta < \delta\Big\} = \sin\frac{1}{n} \simeq \frac{1}{n},$$

Theorem 3.4 yields $\kappa(T_n(a)) = O(n)$ as $n \to \infty$. Notice that this estimate does not depend on v. If $v(\theta) = \cos\theta$ (Example 1.4), then actually $\kappa(T_n(a)) = O(1)$, and $O(n)$ is too crude. However, if $v(\theta) = O(|\theta|)$ as $\theta \to 0$ (which is, for example, the case if $v(\theta) = 0$, $v(\theta) = \sin\theta$, or $v(\theta) = \sin|\theta|$), then Theorem 4.1 will show that the estimate $O(n)$ cannot be improved. $\blacksquare$

4. Lower estimates for the condition numbers

The following theorem provides us with lower asymptotic estimates for the condition numbers of $T_n(a)$ in case a is arbitrary (and not necessarily semi-definite) function in L^∞ which, however, behaves sufficiently well in a vicinity of the zeros.

Theorem 4.1. *Let α, β be positive constants, let $e^{i\theta_0} \in \mathbf{T}$, and suppose $a \in L^\infty$.*

(a) *If $a(e^{i\theta}) = O(|\theta - \theta_0|^\alpha)$ as $\theta \to \theta_0$ then there is a constant $C \in (0, \infty)$ such that*

$$\kappa\Big(T_n(a)\Big) \geq Cn^\alpha \ \text{ for all } \ n \geq 1.$$

(b) *If $a(e^{i\theta}) = O(1/|\log(\theta - \theta_0)/\pi|\,|^\alpha)$ as $\theta \to \theta_0$ then there exists a constant $C \in (0, \infty)$ such that*

$$\kappa\Big(T_n(a)\Big) \geq C(\log n)^\alpha \ \text{ for all } \ n \geq 1.$$

(c) *If $a(e^{i\theta}) = O(e^{-\beta|\theta - \theta_0|^{-\alpha}})$ as $\theta \to \theta_0$ then*

$$\lim_{n\to\infty} n^{-k} \kappa\Big(T_n(a)\Big) = \infty$$

for every $k > 0$.

Recall that we define $\kappa(T_n(a)) = \infty$ in case $T_n(a)$ is not invertible. The proof of Theorem 4.1 will be based on an auxiliary result.

For $j, m \in \mathbf{N}$, consider the trigonometric polynomial

$$p_m^j(\theta) = \Big(1 + e^{i\theta} + \ldots + e^{im\theta}\Big)^j = e^{imj\theta/2} \left(\frac{\sin \frac{m+1}{2}\theta}{\sin \frac{\theta}{2}} \right)^j. \tag{28}$$

Clearly, $p_m^j \in \mathcal{P}_{mj+1}$ and $\|p_m^j\|_\infty = (m+1)^j$. In the case $j = 1$, Parseval's equality gives

$$\|p_m^1\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |p_m^1(\theta)|^2 d\theta = 1^2 + 1^2 + \ldots + 1^2 = m + 1. \tag{29}$$

Lemma 4.2. *For each $j \geq 1$, there exists a constant $D_j \in [1, \infty)$ such that*

$$(1/D_j) m^{2j-1} \leq \|p_m^j\|_2^2 \leq D_j m^{2j-1} \tag{30}$$

for all $m \geq 1$.

Proof. For $j = 1$, this follows from (29). So let $j \geq 2$. Then

$$\begin{aligned} \|p_m^j\|_2^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin((m+1)\theta/2)}{\theta/2} \right)^{2j} \left(\frac{\theta/2}{\sin(\theta/2)} \right)^{2j} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin((m+1)\theta/2)}{\theta/2} \right)^{2j} \Big(1 + O(\theta^2)\Big)\, d\theta \end{aligned} \tag{31}$$

and

$$\int_{-\pi}^{\pi} \left(\frac{\sin((m+1)\theta/2)}{\theta/2} \right)^{2j} \theta^2 d\theta = 8(m+1)^{2j-3} \int_{-\pi(m+1)/2}^{\pi(m+1)/2} \left(\frac{\sin x}{x} \right)^{2j} x^2 dx = O\Big(m^{2j-3}\Big),$$

$$\int_{-\pi}^{\pi} \left(\frac{\sin((m+1)\theta/2)}{\theta/2}\right)^{2j} d\theta = 2(m+1)^{2j-1} \int_{-\pi(m+1)/2}^{\pi(m+1)/2} \left(\frac{\sin x}{x}\right)^{2j} dx$$

$$= 2(m+1)^{2j-1}\Bigg(\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^{2j} dx + o(1)\Bigg),$$

which shows that

$$\lim_{m\to\infty} m^{-(2j-1)}\|p_m^j\|_2^2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^{2j} dx. \blacksquare$$

Actually we will need only the left estimate of (30). Here is another proof of this estimate, which also specifies the constant $1/D_j$.

Lemma 4.3. *For $j, m \in \mathbf{N}$ the inequality*

$$\|p_m^j\|_2^2 > \frac{16}{9\pi}\frac{1}{\sqrt{j}}(m+1)^{2j-1}$$

holds.

Proof. Starting with (31) we obtain

$$2\pi\|p_m^j\|_2^2 > 2\int_0^{\pi} \left(\frac{\sin((m+1)\theta/2)}{\theta/2}\right)^{2j} d\theta$$

$$= 4(m+1)^{2j-1}\int_0^{(m+1)\pi/2} \left(\frac{\sin x}{x}\right)^{2j} dx > 4(m+1)^{2j-1}\int_0^{1/\sqrt{j}} \left(\frac{\sin x}{x}\right)^{2j} dx$$

$$> 4(m+1)^{2j-1}\int_0^{1/\sqrt{j}} \left(1-\frac{x^2}{6}\right)^{2j} dx > 4(m+1)^{2j-1}\int_0^{1/\sqrt{j}} \left(1-\frac{jx^2}{3}\right) dx$$

$$= \frac{32}{9}\frac{1}{\sqrt{j}}(m+1)^{2j-1}. \blacksquare$$

Proof of Theorem 4.1. We first prove part (a). Suppose $\theta_0 = 0$ and let us assume that $|a(e^{i\theta})| \le K|\theta|^\alpha$ for $|\theta| < \delta$. Fix $n > 1/\delta$. We have

$$2\pi\|ap_m^j\|_2^2 = \int_{-\pi}^{\pi} |a(e^{i\theta})|^2\left(\frac{\sin((m+1)\theta/2)}{\sin(\theta/2)}\right)^{2j} d\theta =: \int_{-\pi}^{\pi} f(\theta)\, d\theta \tag{32}$$

for every $m, j \in \mathbf{N}$. Fix any $j \in \mathbf{N}$ such that $j > \alpha + 1/2$. Because $\|p_m^j\|_\infty = (m+1)^j$, we get

$$\int_{-1/m}^{1/m} f(\theta)\, d\theta \le K^2\frac{1}{m^{2\alpha}}(m+1)^{2j}\int_{-1/m}^{1/m} d\theta = K^2\frac{1}{m^{2\alpha}}(m+1)^{2j}\frac{2}{m}$$

and since $|p_m^j(\theta)| < 1/(\sin\frac{\theta}{2})^{2j} < (\pi/\theta)^{2j}$ for $0 < |\theta| < \pi$, we obtain

$$\begin{aligned}
\int\limits_{1/m<|\theta|<\delta} f(\theta)\,d\theta &\le 2K^2 \int\limits_{1/m}^{\delta} \theta^{2\alpha}(\pi/\theta)^{2j} d\theta = 2\pi^{2j}K^2 \int\limits_{1}^{m\delta} \left(\frac{x}{m}\right)^{2(\alpha-j)} \frac{dx}{m} \\
&< 2\pi^{2j}K^2 m^{2j-1} m^{-2\alpha} \int\limits_{1}^{\infty} x^{2(\alpha-j)} dx \le M_1 \frac{m^{2j-1}}{m^{2\alpha}}
\end{aligned}$$

with $M_1 < \infty$ (note that $2(\alpha - j) < -1$). Finally,

$$\int\limits_{\delta<|\theta|<\pi} f(\theta)\,d\theta \le 2\|a\|_\infty^2 \int\limits_{\delta}^{\pi} (\pi/\theta)^{2j} d\theta =: M_2.$$

In summary, there is a constant $M_3 < \infty$ such that

$$\|ap_m^j\|_2^2 \le M_3 \frac{1}{m^{2\alpha}} m^{2j-1}$$

for all $m > 1/\delta$, and Lemma 4.2 (or Lemma 4.3) therefore implies that

$$\|ap_m^j\|_2^2 \le M_3 D_j \frac{1}{m^{2\alpha}} \|p_m^j\|_2^2 =: M_4 \frac{1}{m^{2\alpha}} \|p_m^j\|_2^2. \tag{33}$$

Given n, we write $n = mj + k$ with $k \in \{1, \ldots, j\}$. From (33) we infer that

$$\left\|T_n(a)p_m^j\right\|_2^2 \le \|ap_m^j\|_2^2 \le M_4 \frac{1}{m^{2\alpha}} \|p_m^j\|_2^2 = M_4 \frac{(2j)^{2\alpha}}{(2mj)^{2\alpha}} \|p_m^j\|_2^2 \le M_4 \frac{(2j)^{2\alpha}}{n^{2\alpha}} \|p_m^j\|_2^2,$$

whence $\|T_n^{-1}(a)\| \ge M_4^{-1}(2j)^{-2\alpha} n^{2\alpha}$. The proof of part (a) is complete.

To prove (b) suppose $|a(e^{i\theta})| \le K/|\log(|\theta|/\pi)|^\alpha$ for $|\theta| < \delta$. Put $j = 1$, $m = n$, and define f by (32). Then for sufficiently large n,

$$\int\limits_{-1/n}^{1/n} f(\theta)\,d\theta \le \frac{K^2}{(\log(n/\pi))^{2\alpha}} (n+1)^2 \int\limits_{-1/n}^{1/n} d\theta \le M_5 \frac{n}{(\log n)^{2\alpha}},$$

$$\int\limits_{1/n<|\theta|<\delta} f(\theta)\,d\theta \le 2K^2 \int\limits_{1/n}^{\delta} \frac{1}{|\log(\theta/\pi)|^{2\alpha}} \left(\frac{\pi}{\theta}\right)^2 d\theta = 2K^2\pi \int\limits_{\pi/\delta}^{\pi n} \frac{dx}{(\log x)^{2\alpha}} \le M_6 \frac{n}{(\log n)^{2\alpha}},$$

$$\int\limits_{\delta<|\theta|<\pi} f(\theta)\,d\theta \le 2\|a\|_\infty^2 \int\limits_{\delta}^{\pi} \left(\frac{\pi}{\theta}\right)^2 d\theta = M_7,$$

and consequently, by Lemma 4.2 (or Lemma 4.3),

$$2\pi\|ap_n^1\|_2^2 \le (M_5 + M_6 + M_7) \frac{1}{(\log n)^{2\alpha}} D_1 \|p_n^1\|_2^2,$$

which gives the assertion as in part (a).

Finally, if a is as in part (c) then $a(e^{i\theta}) = O(|\theta - \theta_0|^\alpha)$ as $\theta \to \theta_0$ for every $\alpha > 0$. The assertion is therefore immediate from part (a).

If $\theta_0 \neq 0$, the above arguments work with $p_m^j(\theta)$ replaced by $p_m^j(\theta - \theta_0)$. ■

Corollary 4.4. *Let $a \in L^\infty$, suppose $\operatorname{Re} a \geq 0$ a.e., assume a has exactly N zeros $e^{i\theta_1}, \dots, e^{i\theta_N} \in \mathbf{T}$ and $\operatorname{Re} a$ has no other zeros than those of a. Suppose there are $K \in [1, \infty)$, $0 < \alpha_j \leq \beta_j < \infty$, $\delta \in (0,1)$ such that*

$$(1/K)|\theta - \theta_j|^{\beta_j} \leq \operatorname{Re} a(e^{i\theta}) \leq |a(e^{i\theta})| \leq K|\theta - \theta_j|^{\alpha_j} \tag{34}$$

for $|\theta - \theta_j| < \delta$. Then there is a constant $C \in [1, \infty)$ such that

$$(1/C)n^\alpha \leq \kappa\Big(T_n(a)\Big) \leq Cn^\beta \ \text{ for all } \ n \geq 1 \tag{35}$$

where $\alpha := \max\{\alpha_1, \dots, \alpha_N\}$ and $\beta := \max\{\beta_1, \dots, \beta_N\}$.

Proof. The left inequality of (34) and the definition (20) give $\omega_j(n) \leq Kn^{\beta_j}$, and therefore $\kappa(T_n(a)) \leq C_1 n^\beta$ by virtue of Theorem 3.4. Let $\alpha = \alpha_{j_0}$. Taking into account the right inequality of (34) with $j = j_0$, we obtain from Theorem 4.1(a) that $\kappa(T_n(a)) \geq C_2 n^\alpha$. ■

Combining Theorem 3.4 and Theorem 4.1(b) we see that Corollary 4.4 remains true with (34) and (35) replaced by

$$(1/\kappa)\Big(\log|(\theta - \theta_j)/\pi|\Big)^{-\beta_j} \leq \operatorname{Re} a(e^{i\theta}) \leq |a(e^{i\theta})| \leq \kappa\Big(\log|(\theta - \theta_j)/\pi|\Big)^{-\alpha_j} \tag{36}$$

and

$$(1/C)(\log n)^\alpha \leq \kappa\Big(T_n(a)\Big) \leq C(\log n)^\beta,$$

respectively.

The zeros occuring in the next corollary have been very popular for a long time in connection with the Fisher-Hartwig conjecture for Toeplitz determinants (see, e.g., [4]).

Corollary 4.5. *Let $t_1, \dots, t_n$ be distinct points on $\mathbf{T}$, let $\alpha_1, \dots, \alpha_N$ be positive real numbers, and let $b \in L^\infty$ be a function such that $\operatorname{Re} b \geq \varepsilon > 0$ a.e. on $\mathbf{T}$. Put*

$$a(t) := |t - t_1|^{2\alpha_1} \dots |t - t_N|^{2\alpha_N} b(t), \quad t \in \mathbf{T},$$

and $\alpha := \max\{\alpha_1, \dots, \alpha_N\}$. Then

$$\kappa\Big(T_n(a)\Big) \simeq n^{2\alpha}.$$

Proof. Since $|e^{i\theta} - e^{i\theta_j}| \simeq |\theta - \theta_j|$, this is immediate from Corollary 4.4. ■

Note that if, in addition, b is sufficiently smooth, then for the determinants $\det T_n(a)$ we have

$$\det T_n(a) \sim G(b)^n n^{\alpha_1^2 + \dots + \alpha_N^2} E(a) \ \text{ as } \ n \to \infty$$

where $G(b) = \exp(\log b)_0$ and $E(a)$ is a nonzero constant (see [19] and [4]). This reveals that determinants are much more sensitive to singularities than condition numbers.

Example 4.6. Let

$$T_n(a) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The symbol is $a(e^{i\theta}) = 1 - e^{i\theta}$ and traces out the circle of radius 1 centered at 1. Both a and $\operatorname{Re} a$ have a single zero at $\theta = 0$. Since

$$\operatorname{Re} a(e^{i\theta}) = 1 - \cos\theta \simeq \theta^2, \quad |(a(e^{i\theta})| = (2 - 2\cos\theta)^{1/2} \simeq |\theta|$$

in a neighborhood of $\theta = 0$, Corollary 4.4. gives

$$(1/C)n \le \kappa\Big(T_n(a)\Big) \le Cn^2.$$

As

$$T_n^{-1}(a) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = T_n(p_n^1),$$

we see that $\|T_n^{-1}(a)\| = \|T_n(p_n^1)\| \le \|p_n^1\|_\infty = n + 1$ (taking the Frobenius norm even gives $\|T_n^{-1}(a)\| \le \sqrt{n(n+1)/2}\,$). Thus, the truth is $\kappa(T_n(a)) \simeq n$. ∎

Example 4.7. Consider the Cauchy-Toeplitz matrices

$$T_n(c) = \Big(\frac{1}{\pi(1/2 - j - k)}\Big)_{j,k=0}^{n-1}.$$

We have

$$T_n(c) = \operatorname{diag}\Big((-1)^j\Big) T_n(a) \operatorname{diag}\Big((-1)^k\Big)$$

where

$$T_n(a) = \Big(\frac{(-1)^{j-k}}{\pi(1/2 - j - k)}\Big)_{j,k=0}^{n-1}.$$

Clearly, $\kappa(T_n(c)) = \kappa(T_n(a))$. The symbol $a(e^{i\theta})$ is just the function $e^{i\theta/2}$ ($\theta \in (-\pi, \pi]$) of Example 1.5. The real part $u(\theta) = \cos(\theta/2)$ has a single zero at $\theta = \pi$. Since

$$\frac{1}{\omega(n)} = \min\Big\{\cos\frac{\theta}{2} : \frac{1}{n} < |\theta - \pi| < \delta\Big\} = \sin\frac{1}{2n} \simeq \frac{1}{n}$$

Theorem 3.4 implies that $\kappa(T_n(a)) = O(n)$. Although $\operatorname{Re} a$ has a zero of "order" 1, the symbol a itself has no zero. Therefore Theorem 4.1 is not applicable. In the case at hand we actually have

$$\kappa\Big(T_n(a)\Big) \simeq \log n;$$

this was shown by Tyrtyshnikov [17]. ∎

5. Exponentially growing condition numbers

Theorem 4.1 tells us that if a has a "very strong" zero, like $e^{-1/|\theta-\theta_j|}$ say, then $\kappa(T_n(a))$ increases faster than any polynomial. We are therefore led to the question whether there are symbols $a \in L^\infty \setminus \{0\}$ such that $\kappa(T_n(a))$ grows even faster than $e^{\alpha n}$ $(\alpha > 0)$. The following result shows that this happens if a vanishes on open sets.

Theorem 5.1. *Let $a \in L^\infty \setminus \{0\}$ and suppose $a(e^{i\theta}) = 0$ for $|\theta| < \delta$ where $\delta \in (0, \pi)$. Then*

$$\kappa\Big(T_{n+1}(a)\Big) \geq \frac{\|T_{n+1}(a)\|}{\|a\|_\infty} \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{1}{n^{1/4}} \Big(\frac{1}{\cos(\delta/2)}\Big)^n \tag{37}$$

for all $n \geq 1$ and thus

$$\kappa\Big(T_n(a)\Big) > \frac{1}{2} \frac{1}{n^{1/4}} \Big(\frac{1}{\cos(\delta/2)}\Big)^n$$

for all sufficiently large n.

Proof. Consider the polynomials $p_m^j \in \mathcal{P}_{mj+1}$ given by (28) and put $m = 1$. We have

$$2\pi\|ap_1^j\|_2^2 = 2\int_\delta^\pi |a(e^{i\theta})|^2 \Big(\frac{\sin\theta}{\sin(\theta/2)}\Big)^2 d\theta$$

$$= 2\int_\delta^\pi |a(e^{i\theta})|^2 \Big(2\cos\frac{\theta}{2}\Big)^{2j} d\theta \leq 2^{2j+1}\pi\|a\|_\infty^2 \Big(\cos\frac{\delta}{2}\Big)^{2j}.$$

From Lemma 4.3 we know that

$$\|p_1^j\|_2^2 \geq \frac{16}{9\pi} \frac{1}{\sqrt{j}} 2^{2j-1}.$$

Consequently,

$$\Big\|T_{j+1}(a)p_1^j\Big\|_2^2 \leq \Big\|ap_1^j\Big\|_2^2 \leq \frac{9}{8}\pi\sqrt{j}\|a\|_\infty^2 \Big(\cos\frac{\delta}{2}\Big)^{2j} \|p_1^j\|_2^2.$$

Replacing j by n we obtain (37). ■

Since $r^n n^{-1/4}$ increases faster than s^n where $s \in (1, r)$, we see from Theorem 5.1 that $\kappa(T_n(a))$ increases faster than $e^{\alpha n}$ if only δ is sufficiently close to π.

Theorem 5.2. *Let $a \in L^\infty$ have a zero at $e^{i\theta_0} \in \mathbf{T}$ and suppose $|a(e^{i\theta})| \leq \kappa e^{-1/|\theta-\theta_0|}$ with some $K \in (0, \infty)$ for $|\theta - \theta_0| < \delta$. Then there is a constant $c \in (0, \infty)$ such that*

$$\kappa\Big(T_n(a)\Big) \geq c(e/2)^{\sqrt{n}} n^{-3/4} \text{ for all } n \geq 1. \tag{38}$$

Proof. Without loss of generality assume $\theta_0 = 0$. We start again with (32). For $m > 1/\delta$,

$$\int_{-1/m}^{1/m} f(\theta)\, d\theta \leq 2K^2(m+1)^{2j} \int_0^{1/m} e^{-2/\theta} d\theta$$

and

$$\int_0^{1/m} e^{-2/\theta} d\theta = \int_m^{\infty} \frac{e^{-2x}}{x^2}\, dx \le M_1 \frac{e^{-2m}}{(m+1)^2}.$$

Further,

$$\int_{1/m<|\theta|<\pi} f(\theta)\, d\theta \le 2\|a\|_\infty^2 \int_{1/m}^{\pi} e^{-2/\theta}\Big(\sin(\theta/2)\Big)^{-2j} d\theta. \tag{39}$$

The maximum of the integrand of the last integral on $(0,\pi)$ is attained at the solution θ_j of the equation

$$\frac{1}{j} = \frac{\theta^2}{2}\cot\frac{\theta}{2}. \tag{40}$$

The right-hand side of (40) is $\theta + O(\theta^3)$ as $\theta \to 0$, which implies that

$$\theta_j = \frac{1}{j}\Big(1 + O\Big(\frac{1}{j^2}\Big)\Big) \quad \text{as} \quad j \to \infty.$$

Hence, the maximum of $e^{-2/\theta}(\sin(\theta/2))^{-2j}$ is

$$\begin{aligned} & e^{-2j+O(1/j)}\Big(\sin\Big(\frac{1}{2j} + O\Big(\frac{1}{j^3}\Big)\Big)\Big)^{-2j} \simeq e^{-2j}\Big(\frac{1}{2j} + O\Big(\frac{1}{j^3}\Big)\Big)^{-2j} \\ & = e^{-2j}(2j)^{2j}\Big(1 + O\Big(\frac{1}{j^2}\Big)\Big)^{-2j} \simeq e^{-2j}(2j)^{2j}. \end{aligned}$$

It follows that (39) is not greater than $M_2\, e^{-2j}(2j)^{2j}$ where $M_2 < \infty$ is some constant independent of m and j. In summary,

$$\|ap_m^j\|_2^2 \le 2K^2 M_1\, e^{-2m}(m+1)^{2j-2} + M_2\, e^{-2j}(2j)^{2j}.$$

Taking into account Lemma 4.3, we get

$$\frac{\|ap_m^j\|_2^2}{\|p_m^j\|_2^2} \le M_3\, e^{-2m}\frac{\sqrt{j}}{m+1} + M_4\, \frac{e^{-2j}(2j)^{2j}\sqrt{j}}{(m+1)^{2j-1}}. \tag{41}$$

Now suppose we are given $n > 4/\delta^2$. We put $m = j = [\sqrt{n}\,] - 1$. From (41) we then obtain

$$\begin{aligned} & \frac{\|T_n(a)p_m^j\|_2^2}{\|p_m^j\|_2^2} \le M_3\, e^{-2m}\frac{\sqrt{m}}{m+1} + M_4\, \frac{e^{-2m}2^{2m}\sqrt{m}}{(m+1)^{-1}} \\ & = \Big(\frac{2}{e}\Big)^{2m}(m+1)\sqrt{m}\Big(\frac{M_3}{2^{2m}}\frac{1}{(m+1)^2} + M_4\Big) \le M_5\,\Big(\frac{2}{e}\Big)^{2m} m^{3/2}, \end{aligned}$$

which implies (38). ■

The estimate (38) can certainly be improved but we will not embark on this problem. The conclusion of Theorem 5.2 is that even in the case of a single zero $\kappa(T_n(a))$ may grow faster than $e^{\alpha\sqrt{n}}$.

6. Singular values, eigenvalues, finite section method

Given a selfadjoint $n \times n$ matrix A_n, we denote by

$$\lambda_1(A_n) \le \lambda_2(A_n) \le \ldots \le \lambda_n(A_n) \tag{42}$$

the eigenvalues of A_n. The *singular values* $s_j(A_n)$ of an arbitrary $n \times n$ matrix A_n are the nonnegative square roots of the eigenvalues of the selfadjoint and positive semi-definite matrix $A_n^* A_n$, i.e.

$$s_j(A_n) = \sqrt{\lambda_j(A_n^* A_n)}.$$

In accordance with the ordering (42),

$$0 \le s_1(A_n) \le s_2(A_n) \le \ldots \le s_n(A_n).$$

It is easily seen that

$$s_1(A_n) = 1/\|A_n^{-1}\| \ \text{ and } \ s_n(A_n) = \|A_n\|.$$

Hence, $\kappa(A_n) = s_n(A_n)/s_1(A_n)$. If A_n is selfadjoint, then $\lambda_j(A_n) = s_j(A_n)$ for all j and $\kappa(A_n) = \lambda_n(A_n)/\lambda_1(A_n)$. The results of the preceding sections can therefore also be stated in terms of the singular values and the eigenvalues.

Asymptotics of singular values and eigenvalues of Toeplitz matrices have been studied by many authors; see, e.g., the recent papers [3], [7], [13], [14], [15], [18] and the literature listed there. In particular, the *Avram-Parter theorem* says that if $a \in L^\infty$ then the singular values $\{s_j(T_n(a))\}_{j=1}^n$ and the values $\{|a(e^{2\pi i j/n})|\}_{j=1}^n$ are asymptotically equally distributed in the sense that, for every $F \in C_0^\infty(\mathbf{R})$,

$$\frac{1}{n}\sum_{j=1}^{n} F\Big(s_j\big(T_n(a)\big)\Big) \to \frac{1}{2\pi}\int_0^{2\pi} F\big(|a(e^{i\theta})|\big)\, d\theta$$

as $n \to \infty$. This theorem gives us the hint that if, for example, $a(\theta) = \theta^\alpha$ for $\theta \in (-\pi, \pi]$, then $s_j(T_n(a))$ should decay as $1/n^\alpha$ for each j, but the theorem does not imply that $s_1(T_n(a)) \simeq 1/n^\alpha$.

We now quote three sample results which are immediate consequences of the results of Sections 3 and 4.

Corollary 6.1. *Under the hypotheses of Corollary* 4.4 *we have*

$$\frac{1}{D}\frac{1}{n^\beta} \le s_1\big(T_n(a)\big) \le D\frac{1}{n^\alpha} \ \textit{ for all } \ n \ge 1$$

with some constant $D \in [1, \infty)$. ∎

Obviously, replacing (34) by (36) gives

$$\frac{1}{D}\frac{1}{(\log n)^\beta} \le s_1\big(T_n(a)\big) \le D\frac{1}{(\log n)^\alpha} \ \text{ for all } \ n \ge 1,$$

while the presence of "very strong" zeros, such as $e^{-1/|\theta-\theta_j|}$, yield that $s_1(T_n(a))$ decreases faster than any negative power of n.

Corollary 6.2. *Let $a \in L^\infty$ be real-valued and put $m := \operatorname{ess\,inf} a$. If $a - m$ has exactly N zeros $e^{i\theta_1}, \ldots, e^{i\theta_N} \in \mathbf{T}$ such that*

$$(1/K)|\theta - \theta_j|^{\alpha_j} \le |a(e^{i\theta}) - m| \le K|\theta - \theta_j|^{\alpha_j}$$

for $|\theta - \theta_j| < \delta$ with $K \in [1, \infty)$ and $\alpha_j \in (0, \infty)$, then there is a constant $D \in [1, \infty)$ such that

$$\frac{1}{D}\frac{1}{n^\alpha} \le \lambda_1\Big(T_n(a)\Big) - m \le D\frac{1}{n^\alpha} \text{ for all } n \ge 1.$$

where $\alpha := \max\{\alpha_1, \ldots, \alpha_N\}$. ■

In the case $N = 1$ and $\alpha_j = 2$, this result is essentially already in [8, Example 5.3].

Corollary 6.3. *Let $T(a)$ be a selfadjoint Toeplitz band matrix and put $m = \min a$. If $T(a)$ is not a constant multiple of the identity matrix, then there is a constant $C \in (0, \infty)$ such that*

$$0 < \lambda_1\Big(T_n(a)\Big) - m < C\frac{1}{n^2} \text{ for all } n \ge 1.$$

In other words, $\kappa(T_n(a) - mI)$ increases at least as n^2 to infinity.

Proof. We can write

$$a(e^{i\theta}) - m = \frac{a_0}{2} + \sum_{k=0}^{M}(a_k \cos k\theta + b_k \sin k\theta), \quad \theta \in (-\pi, \pi].$$

The function $a(e^{i\theta}) - m$ has at least one zero $\theta_0 \in (-\pi, \pi]$. Since $a(e^{i\theta}) - m$ is nonnegative, the Taylor expansion at θ_0 reads

$$a(e^{i\theta}) - m = A(\theta - \theta_j)^2 + O\Big((\theta - \theta_0)^2\Big)$$

with $A \ge 0$. Consequently, $a(e^{i\theta}) - m$ has a zero of order at least 2, and Theorem 4.1(a) therefore gives the assertion. ■

Obviously, the conclusion of Corollary 6.3 remains true for selfadjoint Toeplitz matrices with twice continuously differentiable symbols.

The *finite section method* for solving the infinite system $T(a)x = y$ consists in passage to the truncated systems

$$T_n(a)x^{(n)} = P_n y, \quad x^{(n)} \in \operatorname{Im} P_n \tag{43}$$

where $P_n : l^2 \to l^2$ is the projection defined by

$$P_n : \{x_0, x_1, x_2, \ldots\} \mapsto \{x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots\}.$$

If $T(a)$ is invertible, then the finite section method is applicable in many cases (see, e.g., [5] and [4]). It is in particular applicable if $T(a)$ is positive definite. Things are more complicated if $T(a)$ is merely known to be semi-definite.

Some results are available provided we can estimate the growth of $\|T_n^{-1}(a)\|$. The following simple proposition is well known (see, e.g., [12]) and is merely cited in order to illustrate the usefulness of such results as Theorem 3.4.

Let l^2_α $(\alpha > 0)$ denote the Hilbert space of all complex sequences $\{x_n\}_{n=0}^\infty$ such that $\sum (n+1)^{2\alpha}|x_n|^2 < \infty$.

Proposition 6.4. *Let $a \in L^\infty$ and suppose $\operatorname{Re} a \geq 0$ a.e. If*

$$\kappa\Big(T_n(a)\Big) = O(n^\alpha) \quad (\alpha \geq 0) \tag{44}$$

and y is an element of l^2 such that the equation $T(a)x = y$ has a solution $x \in l^2_\alpha$ then the solutions $x^{(n)}$ of (43) converge to x in the norm of l^2.

Proof. We observed in the introduction that $T(a)$ is injective and that $T_n(a)$ is invertible for all $n \geq 1$. Let $T_n(a)x^{(n)} = P_n y$. We have

$$\|x^{(n)} - x\| \leq \|T_n^{-1}(a)P_n y - P_n x\| + \|P_n x - x\|$$

and it is clear that $\|P_n x - x\| \to 0$. By (44),

$$\begin{aligned} &\|T_n^{-1}(a)P_n y - P_n x\| \leq C n^\alpha \|P_n y - T_n(a)P_n x\| \\ &\leq C n^\alpha \|P_n y - P_n T(a)x\| + C n^\alpha \|P_n T(a) Q_n x\| \end{aligned} \tag{45}$$

where $Q_n := I - P_n$. The first term in (45) is zero and the second term is at most

$$\begin{aligned} &C n^\alpha \|a\|_\infty \|Q_n x\| = C n^\alpha \|a\|_\infty \Big(\sum_{j \geq n} |x_j|^2\Big)^{1/2} \\ &\leq C n^\alpha \|a\|_\infty \frac{1}{n^\alpha}\Big(\sum_{j\geq n}(j+1)^{2\alpha}|x_j|^2\Big)^{1/2} = C\|a\|_\infty \|Q_n x\|_{l^2_\alpha}. \end{aligned}$$

Since $\|Q_n x\|_{l^2_\alpha} \to 0$ whenever $x \in l^2_\alpha$, it follows that $\|x^{(n)} - x\| \to 0$. ∎

7. Finite Wiener-Hopf integral operators

Let $\tau \in (0,\infty)$ and $a \in L^\infty(\mathbf{R})$. The Wiener-Hopf integral operator $W_\tau(a)$ is defined on $L^2(0,\tau)$ by the formula

$$\Big(W_\tau(a)f\Big)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(a(\xi)\int_0^\tau f(y)e^{i\xi y}dy\Big)e^{-i\xi x}dx, \quad x \in (0,\tau).$$

We remark that if

$$a(\xi) = c + \int_{-\infty}^{\infty} k(x)e^{i\xi x}dx, \quad \xi \in \mathbf{R},$$

with some function $k \in L^1(\mathbf{R})$, then $W_\tau(a)$ acts by the rule

$$\Big(W_\tau(a)f\Big)(x) = cf(x) + \int_0^\tau k(x-t)f(t)\,dt, \quad x \in (0,\tau).$$

To have another example, note that in the case where $a(\xi) = \operatorname{sign} \xi$ we have

$$\Big(W_\tau(a)f\Big)(x) = \frac{1}{\pi i} \int_0^\tau \frac{f(t)}{x-t}\, dt, \quad x \in (0,\tau),$$

the integral understood in the Cauchy principal sense.

Wiener-Hopf integral operators are the continuous analogues of Toeplitz matrices. The results of the previous sections can be easily extended to Wiener-Hopf operators. The part of $e^{in\theta}$ and $\cos n\theta$ is now played by $e^{i\tau\xi}$ and $\cos \tau\xi$. Notice that

$$W_\tau(a) = W_\tau(a + ig_\sigma) \text{ where } g_\sigma(\xi) := \cos \sigma\xi \text{ and } \sigma \geq \tau.$$

Instead of the polynomials $p_n^j(e^{i\theta})$ employed in Section 4 one can work with the functions

$$\varphi_\tau^j(\xi) = \Big(\int_0^\tau e^{i\xi x}\, dx\Big)^j = \Big(\frac{e^{i\tau\xi} - 1}{i\xi}\Big)^j, \quad \xi \in \mathbf{R}. \tag{46}$$

Note that

$$\varphi_\tau^j(\xi) = e^{ij\tau\xi/2}\Big(\frac{\sin\frac{\xi\tau}{2}}{\xi/2}\Big)^j, \quad \xi \in \mathbf{R}, \tag{47}$$

and that there are constants $D_j \in [1,\infty)$ such that

$$(1/D_j)\tau^{j-1/2} \leq \|\varphi_\tau^j\|_{L^2(\mathbf{R})} \leq D_j \tau^{j-1/2}$$

for all $\tau > 0$ and all $j \in \mathbf{N}$.

Here are the Wiener-Hopf analogous of Theorems 3.4 and 4.1.

Theorem 7.1. *Let $a \in L^\infty(\mathbf{R})$, suppose $\operatorname{Re} a \geq 0$ a.e. on $\mathbf{R}$, and assume $u := \operatorname{Re} a$ has exactly $N \geq 1$ zeros $\xi_1, \ldots, \xi_N$ on $\mathbf{R}$. Let*

$$\operatorname{ess\,inf}\Big\{u(\xi) : \xi \in \mathbf{R} \setminus \bigcup_{j=1}^N (\xi_j - \delta, \xi_j + \delta)\Big\} > 0,$$

put

$$\frac{1}{\omega_j(\tau)} := \operatorname{ess\,inf}\Big\{u(\xi) : \frac{1}{\tau} < |\xi - \xi_j| < \delta\Big\},$$

and define $\omega(\tau) := \max\{\omega_1(\tau), \ldots, \omega_N(\tau)\}$. Then $W_\tau(a)$ is invertible for all $\tau > 0$ and

$$\kappa\Big(W_\tau(a)\Big) = O\Big(\omega(13^{N+1}\tau)\Big) \text{ as } \tau \to \infty. \blacksquare$$

Theorem 7.2. *Let α, β, τ_0 be positive constants, let $\xi_0 \in \mathbf{R}$, and suppose $a \in L^\infty(\mathbf{R})$.*

(a) *If $|a(\xi)| = O(|\xi - \xi_0|^\alpha)$ as $\xi \to \xi_0$ then there is a constant $C \in (0,\infty)$ such that*

$$\kappa\Big(W_\tau(a)\Big) \geq C\tau^\alpha \text{ for all } \tau \geq \tau_0.$$

(b) *If* $|a(\xi)| = O(1/|\log|\xi-\xi_0|\,|^\alpha)$ *as* $\xi \to \xi_0$ *then there exists a constant* $C \in (0,\infty)$ *such that*

$$\kappa\Big(W_\tau(a)\Big) \ge C\Big(\log\frac{\tau}{\tau_0}\Big)^\alpha \text{ for all } \tau \ge \tau_0.$$

(c) *If* $|a(\xi)| = O(e^{-\beta|\xi-\xi_0|^{-\alpha}})$ *as* $\xi \to \xi_0$ *then*

$$\lim_{\tau\to\infty} \tau^{-k}\kappa\Big(W_\tau(a)\Big) = \infty$$

for every $k > 0$. ■

Corollary 7.3. *Let* $\xi_1,\dots,\xi_N$ *be distinct points on* $\mathbf{R}$, *let* $\alpha_1,\dots,\alpha_N$ *be positive real numbers, and let* $b \in L^\infty(\mathbf{R})$ *be a function for which* $\operatorname{Re} b \ge \varepsilon > 0$ *a.e. on* $\mathbf{R}$. *Put*

$$a(\xi) = \left|\frac{\xi-\xi_1}{\xi+i}\right|^{2\alpha_1}\dots\left|\frac{\xi-\xi_N}{\xi+i}\right|^{2\alpha_N} b(\xi), \quad \xi\in\mathbf{R},$$

and define $\alpha := \max\{\alpha_1,\dots,\alpha_N\}$. *Then for every* $\tau_0 > 0$ *there exists a constant* $C(\tau_0) \in [1,\infty)$ *such that*

$$\Big(1/C(\tau_0)\Big)\tau^{2\alpha} \le \kappa\Big(W_\tau(a)\Big) \le C(\tau_0)\tau^{2\alpha} \textit{ for all } \tau \ge \tau_0. \blacksquare$$

Here is a Wiener-Hopf analogue of Theorem 5.1.

Theorem 7.4. *Let* $a \in L^\infty(\mathbf{R}) \setminus \{0\}$ *and suppose* $a(\xi) = 0$ *for* $|\xi| > \delta$ *where* $\delta \in (0,\infty)$. *Then*

$$\kappa\Big(W_\tau(a)\Big) > \frac{\|W_\tau(a)\|}{\|a\|_\infty}\frac{2}{3\sqrt{\pi}}\Big(\frac{\delta}{4e^3}\Big)^{1/4}\tau^{1/4}\Big(e^{\delta/(4e)}\Big)^\tau$$

for $\tau > 8e/\delta$ *and thus*

$$\kappa\Big(W_\tau(a)\Big) > \frac{1}{2\sqrt{\pi}}\Big(\frac{\delta}{4e^3}\Big)^{1/4}\tau^{1/4}\Big(e^{\delta/(4e)}\Big)^\tau$$

for all sufficiently large τ.

Proof. Let φ_τ^j be the function (46). One can show as in the proof of Lemma 4.3 that

$$\|\varphi_\tau^j\|_2^2 > \frac{16}{9\pi}\frac{1}{\sqrt{j}}\tau^{2j-1}.$$

If $\tau > 0$ and $j \in \mathbf{N}$, then, by (47),

$$\frac{\|a\varphi_{\tau/j}^j\|_2^2}{\|\varphi_{\tau/j}^j\|_2^2} \le 2\|a\|_\infty^2 \int_\delta^\infty \Big(\frac{2}{\xi}\Big)^{2j} d\xi\, \frac{9\pi}{16}\sqrt{j}\Big(\frac{j}{\tau}\Big)^{2j-1} = \frac{9\pi}{4}\|a\|_\infty^2 \frac{\sqrt{j}}{2j-1}\Big(\frac{2j}{\delta\tau}\Big)^{2j-1}.$$

Letting $j = [\tau]+1$ we therefore get

$$\|W_\tau^{-1}(a)\| \ge \frac{2}{3\sqrt{\pi}}\frac{1}{\|a\|_\infty}\frac{(2[\tau]+1)^{1/2}}{([\tau]+1)^{1/4}}\Big(\frac{\delta\tau}{2([\tau]+1)}\Big)^{[\tau]+1/2}. \tag{48}$$

This shows that $\|W_\tau^{-1}(a)\|$ increases exponentially if $\delta > 2$. To cover the case $\delta \leq 2$, we proceed as follows. For $\lambda > 0$, consider the invertible isometry

$$U_\lambda : L^2(0,\tau) \to L^2(0,\tau/\lambda),\ (U_\lambda f)(x) = \sqrt{\lambda} f(\lambda x).$$

A straightforward computation gives

$$U_\lambda W_\tau(a) U_\lambda^{-1} = W_{\tau/\lambda}(a_\lambda) \ \text{ where } \ a_\lambda(\xi) = a(\xi/\lambda).$$

Applying (48) to $W_{\tau/\lambda}(a_\lambda)$ we get

$$\|W_\tau^{-1}(a)\| = \|W_{\tau/\lambda}^{-1}(a_\lambda)\| \geq \frac{2}{3\sqrt{\pi}} \frac{1}{\|a\|_\infty} \frac{(2[\tau/\lambda]+1)^{1/2}}{([\tau/\lambda]+1)^{1/4}} \Big(\frac{\lambda\delta\tau/\lambda}{2([\tau/\lambda]+1)} \Big)^{[\tau/\lambda]+1/2} \tag{49}$$

(note that $a_\lambda(\xi)$ vanishes for $|\xi| < \lambda\delta$). The function $(\lambda\delta/4)^{1/\lambda}$ attains its maximum at $4e/\delta$. Thus, let us put $\lambda = 4e/\delta$ in (49) and let us assume that $\tau > 2\lambda = 8e/\delta$. Then

$$\begin{gathered} \frac{(2[\tau/\lambda]+1)^{1/2}}{([\tau/\lambda]+1)^{1/4}} \geq \frac{(2[\tau/\lambda])^{1/2}}{(2[\tau/\lambda])^{1/4}} > 2^{1/4} \Big(\frac{\tau}{2\lambda}\Big)^{1/4} = \Big(\frac{\tau}{\lambda}\Big)^{1/4}, \\ \frac{\lambda\delta\tau/\lambda}{2([\tau/\lambda]+1)} > \frac{\lambda\delta}{4} = e, \ \ \Big[\frac{\tau}{\lambda}\Big] + \frac{1}{2} > \frac{\tau}{\lambda} - \frac{1}{2}, \end{gathered}$$

which implies the assertion. ∎

Under the hypotheses of Theorem 7.1 and Corollary 7.3, the operators $W_\tau(a)$ are invertible for all $\tau > 0$. This is no longer true if the symbol a has a zero at infinity.

Proposition 7.5. *Let $a \in L^\infty(\mathbf{R})$ and suppose*

$$\lim_{\eta\to\infty} \operatorname*{ess\,sup}_{\xi>\eta} |a(\xi)| = 0.$$

Then $W_\tau(a)$ is not invertible and thus $\kappa(W_\tau(a)) = \infty$ for all $\tau > 0$.

Proof. Fix $\tau > 0$, put $\xi_n := \pi n/\tau$, and define $\psi_n \in L^2(0,\tau)$ by

$$\psi_n(x) = \frac{1}{\sqrt{\tau}} e^{-2i\xi_n x}, \ \ 0 < x < \tau.$$

Clearly, $\|\psi_n\|_2 = 1$. The Fourier transform $\hat{\psi}_n$ of ψ_n is

$$\hat{\psi}_n(\xi) = \int_0^\tau \frac{1}{\sqrt{\tau}} e^{ix(\xi-2\xi_n)} dx = -\frac{i}{\sqrt{\tau}} \frac{e^{i\xi\tau}-1}{\xi - 2\xi_n}.$$

Hence

$$\begin{aligned} 2\pi\|W_\tau(a)\psi_n\|_2^2 &\leq \int_{-\infty}^{\infty} |a(\xi)\hat{\psi}_n(\xi)|^2 d\xi \\ &\leq \|a\|^2_{L^\infty(\xi_n,3\xi_n)} \|\hat{\psi}_n\|_2^2 + \|a\|_\infty^2 \int_{|\xi-2\xi_n|>\xi_n} \frac{4d\xi}{\tau|\xi-2\xi_n|^2}. \end{aligned}$$

Since $\|\hat{\psi}_n\|_2^2 = 2\pi$, $\|a\|_{L^\infty(\xi_n,3\xi_n)} \to 0$ as $n \to \infty$, and

$$\int\limits_{|\xi-2\xi_n|>\xi_n} \frac{4d\xi}{\tau|\xi-2\xi_n|^2} = \frac{8}{\tau}\int\limits_{\xi_n}^{\infty} \frac{dr}{r^2} = \frac{8}{\tau\xi_n} = \frac{8}{\tau}\frac{\tau}{\pi n} = \frac{8}{\pi n} \to 0 \text{ as } n \to \infty,$$

it follows that $\|W_\tau(a)\psi_n\|_2 \to 0$. ■

8. Multidimensional Toeplitz matrices

Let $\Omega \in \mathbf{R}^2$ be a bounded set whose set of inner points is not empty. Given a function $a \in L^\infty$ on the torus $\mathbf{T}^2$, the *two-dimensional Toeplitz operator* $T_{n\Omega}(a)$ is defined on $l^2(n\Omega \cap \mathbf{Z}^2)$ by

$$\Big(T_{n\Omega}(a)\varphi\Big)_j = \sum_{k\in n\Omega\cap\mathbf{Z}^2} a_{j-k}\varphi_k, \;\; j \in n\Omega \cap \mathbf{Z}^2,$$

where

$$a_{l,m} := \frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} a(e^{ix}, e^{iy})e^{-ilx}e^{-imy}dx\,dy, \;\; (l,m) \in \mathbf{Z}^2.$$

If $\Omega = [0,1]^2$, then $T_{n\Omega}(a)$ is also referred to as a "two-level" Toeplitz operator, because $T_{n\Omega}(a)$ is in a natural manner unitarily equivalent to a matrix of the form

$$\begin{pmatrix} T_n(b_0) & T_n(b_{-1}) & \dots & T_n(b_{-(n-1)}) \\ T_n(b_1) & T_n(b_0) & \dots & T_n(b_{-(n-2)}) \\ \vdots & \vdots & \ddots & \vdots \\ T_n(b_{n-1}) & T_n(b_{n-2}) & \dots & T_n(b_0) \end{pmatrix}.$$

In the case where Ω is a polygon, criteria for the boundedeness of the condition numbers $\kappa(T_{n\Omega}(a))$ are known (see [11] and [4]). If Ω is arbitrary and $\mathrm{Re}\, a \ge \varepsilon > 0$ a.e. on $\mathbf{T}^2$, then $T_{n\Omega}(a)$ is invertible on $l^2(n\Omega \cap \mathbf{Z}^2)$ for all $n \ge 1$ and $\kappa(T_{n\Omega}(a)) < 2\|a\|_\infty/\varepsilon$. Difficulties arise as soon as only $\mathrm{Re}\, a \ge 0$ is required, mainly because the set of zeros of a function of two variables is typically not discrete. However, in case $\mathrm{Re}\, a$ has merely a finite number of zeros, we can proceed as in Section 3.

Let $a \in L^\infty(\mathbf{T}^2)$ and $\mathrm{Re}\, a \ge 0$ a.e. on $\mathbf{T}^2$. Put $u(x,y) := \mathrm{Re}\, a(e^{ix}, e^{i,y})$. For $(x_0,y_0) \in \mathbf{R}^2$, let

$$U_\delta(x_0,y_0) := \Big\{(x,y) \in \mathbf{R}^2 : (x-x_0)^2 + (y-y_0)^2 < \delta^2\Big\}.$$

A point $(x_0,y_0) \in (-\pi,\pi]^2$ is called a zero of u if

$$\operatorname{ess\,inf}\Big\{u(x,y) : (x,y) \in U_\delta(x_0,y_0)\Big\} = 0$$

for every $\delta > 0$. Assume u has exactly $N \ge 1$ zeros $(x_1,y_1),\dots,(x_N,y_N)$ in $(-\pi,\pi]^2$. Choose $\delta > 0$ so that the disks $U_\delta(x_j,y_j)$ are pairwise disjoint, put

$$\frac{1}{\omega_j(n)} := \operatorname{ess\,inf}\Big\{u(x,y) : \frac{1}{n^2} < (x-x_j)^2 + (y-y_j)^2 < \delta^2\Big\}, \tag{50}$$

$$\omega(n) := \max\Big\{\omega_1(n),\dots,\omega_N(n)\Big\}. \tag{51}$$

Finally, let

$$D_\Omega := \sup\left\{\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} : (x_1,y_1)\in\Omega,\ \ (x_2,y_2)\in\Omega\right\}.$$

Theorem 8.1. *Let $a \in L^\infty(\mathbf{T}^2)$, suppose $u := \operatorname{Re} a \geq 0$ a.e., and assume u has exactly $N \geq 1$ zeros $(x_1,y_1),\ldots,(x_N,y_N)$ in $(-\pi,\pi]^2$. Define $\omega(n)$ by* (50) *and* (51). *Then*

$$\kappa\Big(T_{n\Omega}(a)\Big) \leq 20\Big(\|\operatorname{Im} a\|_\infty + 1\Big)\omega(4D_\Omega 13^{2N} n)$$

for all sufficiently large n.

Proof outline. The proof is similar to the proof of Theorem 3.4. First, it is easily seen that

$$n\Omega + (-n\Omega) \subset [-s_n, s_n]^2 \tag{52}$$

where $s_n = 2nD_\Omega$. Thus, $T_{n\Omega}(a)$ contains only the Fourier coefficients a_{lm} for which $|l| \leq s_n$ and $|m| \leq s_n$. Application of Lemma 3.5 with $\mu = 1/12$ and the $2N$ numbers

$$s_n x_1/(2\pi), \ldots, s_n x_N/(2\pi), s_n y_1/(2\pi), \ldots, s_n y_N/(2\pi)$$

gives an integer q_n such that

$$1 \leq q_n \leq 13^{2N} \ \text{ and } \ s_n q_n x_j,\ s_n q_n y_j \in 2\pi\mathbf{Z} + \Big(-\frac{\pi}{6}, \frac{\pi}{6}\Big).$$

As in the proof of Theorem 3.4 we obtain that

$$\cos(s_n q_n x) > 1/2, \ \ \cos(s_n q_n y) > 1/2$$

whenever $|x - x_j| < r_n$, $|y - y_j| < r_n$ with $r_n := \pi/(6 s_n q_n)$. Denote by v the imaginary part of a, put

$$\frac{1}{\varepsilon_{n,j}} := 5\Big(\|v\|_\infty + 1\Big)\omega_j\Big(\frac{6 s_n q_n}{\pi}\Big), \quad M := 4\Big(\|v\|_\infty + 1\Big),$$

and consider the function

$$b_n(e^{ix}, e^{iy}) := a(e^{ix}, e^{iy}) + iM\cos(s_n q_n x)\cos(s_n q_n y).$$

Then $T_{n\Omega}(a) = T_{n\Omega}(b_n)$. Repeating the argument of the proof of Theorem 3.4 we arrive at the conclusion that the essential range of b_n lies above the straight line $y = 1 - (1/\varepsilon_n)x$ where $\varepsilon_n := \min_j \varepsilon_{n,j}$ if only n is sufficiently large, whence $\|T_{n\Omega}^{-1}(a)\| < 4/\varepsilon_n$. This implies the assertion. ■

Example 8.2. Let $a(e^{ix}, e^{iy}) = 2 - \cos x - \cos y$. The zero of this symbol is $(x,y) = (0,0)$, we have

$$\begin{aligned}\frac{1}{\omega(n)} &= \min\{2 - \cos x - \cos y : x^2 + y^2 \geq 1/n^2\} \\ &= 2\Big(1 - \cos\frac{\sqrt{2}}{2n}\Big) = 4\sin^2\frac{\sqrt{2}}{4n} \simeq \frac{1}{n^2}\end{aligned}$$

and thus, $\kappa(T_{n\Omega}(a)) = O(n^2)$ as $n \to \infty$. ■

Theorem 8.1 generalizes to d-dimensional Toeplitz operators. With $\omega(n)$ and D_Ω defined in the obvious manner, we have

$$\kappa\Big(T_{n\Omega}(a)\Big) = O\Big(\omega(4D_\Omega 13^{dN} n)\Big) \ \text{ as } \ n \to \infty. \tag{53}$$

Note that if $\Omega = [0,1]^d$, then (52) is true with $s_n = n$, so that in (53) the factor $4D_\Omega 13^{dN}$ can be replaced by $2 \cdot 13^{dN}$. Since $2 \cdot 13^{dN} < 13^{dN+1}$, this agrees with Theorem 3.4 in the case $d = 1$.

The extension of Theorem 4.1 to higher dimensions seems to be difficult. The following result is the d-dimensional analogue of Theorem 5.1.

Theorem 8.3. *Suppose $\Omega = [0,1]^d$ and $a \in L^\infty(\mathbf{T}^d) \setminus \{0\}$ vanishes on $(-\delta,\delta)^d$ where $\delta \in (0,\pi)$. Then*

$$\kappa\Big(T_{(n+1)\Omega}(a)\Big) \geq \frac{\|T_{(n+1)\Omega}(a)\|}{\sqrt{2d}\|a\|_\infty}\Big(\frac{8}{9\pi}\Big)^{d/2}\frac{1}{n^{d/4}}\Big(\frac{1}{\cos(\delta/2)}\Big)^n$$

for all $n \geq 1$ and thus

$$\kappa\Big(T_{n\Omega}(a)\Big) > \frac{1}{2\sqrt{d}}\Big(\frac{8}{9\pi}\Big)^{d/2}\frac{1}{n^{d/4}}\Big(\frac{1}{\cos(\delta/2)}\Big)^n$$

for all suffuiciently large n.

Proof outline. To simplify notation let $d = 2$. Put

$$\begin{aligned} R_\delta &= \Big\{(x,y) \in (-\pi,\pi]^2 : |x| \geq \delta \ \text{ or } \ |y| \geq \delta\Big\}, \\ S_\delta &= \Big\{(x,y) \in (-\pi,\pi]^2 : |x| \geq \delta\Big\}. \end{aligned}$$

With p_m^j as in Section 4, we have

$$\begin{aligned} &4\pi^2\Big\|a\Big(p_1^j \otimes p_1^j\Big)\Big\|_2^2 \leq \|a\|_\infty^2 \iint_{R_\delta}\Big(\frac{\sin x}{\sin(x/2)}\Big)^{2j}\Big(\frac{\sin y}{\sin(y/2)}\Big)^{2j} dx\,dy \\ &\leq 2\|a\|_\infty^2 \iint_{S_\delta}\Big(2\cos\frac{x}{2}\Big)^{2j}\Big(2\cos\frac{y}{2}\Big)^{2j} dx\,dy \\ &\leq 2^{4j+1}\|a\|_\infty^2\Big(\cos\frac{\delta}{2}\Big)^{2j}(2\pi)^2. \end{aligned}$$

Since

$$\Big\|p_1^j \otimes p_1^j\Big\|_2^2 \geq \Big(\frac{8}{9\pi}\Big)^2 \frac{1}{j}\, 2^{4j}.$$

by virtue of Lemma 4.3, it follows that

$$\Big\|a\Big(p_1^j \otimes p_1^j\Big)\Big\|_2^2 \leq 2\|a\|_\infty^2\Big(\frac{9\pi}{8}\Big)^2 j\Big(\cos\frac{\delta}{2}\Big)^{2j}\Big\|p_1^j \otimes p_1^j\Big\|_2^2.$$

This gives the assertion as in the proof of Theorem 5.1. ■

9. Approximation by Toeplitz band matrices

For $a \in L^\infty := L^\infty(\mathbf{T})$, denote by $s_n a$ and $\sigma_n a$ the nth partial sum of the Fourier series and the nth Fejer-Cesaro mean, respectively. Thus,

$$(s_n a)(e^{i\theta}) = \sum_{k=-n}^{n} a_n e^{in\theta},$$
$$(\sigma_n a)(e^{i\theta}) = \frac{1}{n+1}\sum_{j=0}^{n}(s_j a)(e^{i\theta}).$$

Clearly, $T(s_n a)$ and $T(\sigma_n a)$ are Toeplitz band matrices.

Suppose a is smooth, say Hölder continuous. Then both $s_n a$ and $\sigma_n a$ converge uniformly to a. Hence, if $T(a)$ is invertible, then $T(s_n a)$ and $T(\sigma_n a)$ are invertible for all sufficiently large n and

$$\|T^{-1}(s_n a) - T^{-1}(a)\| \to 0, \quad \|T^{-1}(\sigma_n a) - T^{-1}(a)\| \to 0,$$

which implies that

$$\lim_{n\to\infty} \kappa\Big(T(s_n a)\Big) = \lim_{n\to\infty} \kappa\Big(T(\sigma_n a)\Big) = \kappa\Big(T(a)\Big). \tag{54}$$

In particular, if $\operatorname{Re} a \geq \varepsilon > 0$ then (54) holds.

Things change if a is smooth and all we know is that $\operatorname{Re} a \geq 0$ on $\mathbf{T}$. Assume, for example, $a(e^{i\theta}) = \theta^2/4$ for $\theta \in (-\pi, \pi]$. The Fourier series is

$$a(e^{i\theta}) = \frac{\pi^2}{12} - \cos\theta + \frac{1}{2^2}\cos 2\theta - \frac{1}{3^2}\cos 3\theta + - \dots$$

and hence

$$(s_n a)(1) = \frac{\pi^2}{12} - 1 + \frac{1}{2^2} - \frac{1}{3^2} + - \dots - \frac{1}{n^2} < 0$$

if n is odd. Consequently, $T(s_n a)$ is not invertible and thus $\kappa(T(s_n a)) = \infty$ for every odd number n. It follows that there is no sequence $\{\omega(n)\}_{n=1}^\infty$ such that $\kappa(T(s_n a)) = O(\omega(n))$ as $n \to \infty$.

The operators $T(\sigma_n a)$ behave much better than the operators $T(s_n a)$. We remark that for large classes of symbols a, e.g., for

$$a \in (C + H^\infty) \cup (C + \overline{H^\infty}) \cup PQC,$$

the operator $T(a)$ is Fredholm of index k if and only if $\sigma_n a$ is bounded away from zero and has the winding number $-k$ for all sufficiently large n (see [4]).

If $a \in L^\infty$ and $\kappa(T(\sigma_n a)) = O(1)$ as $n \to \infty$, then $T(a)$ must be invertible (this follows as in the proof of Proposition 1.2 from the fact that $T(\sigma_n a)$ converges strongly to $T(a)$). Hence, $\kappa(T(\sigma_n a))$ is necessarily unbounded in case a has zeros on $\mathbf{T}$. The following theorem provides us with upper bounds for $\kappa(T(\sigma_n a))$.

Theorem. *Let $a \in L^\infty$, suppose $u(\theta) := \operatorname{Re} a(e^{i\theta})$ is nonnegative and has exactly $N \geq 1$ zeros $\theta_1, \dots, \theta_N$ in $(-\pi, \pi]$, define $\omega_j(n)$ by* (20)*, and put $\omega(n) := \max\{\omega_1(n), \dots, \omega_N(n)\}$. Then*

$$\kappa\Big(T(\sigma_n a)\Big) \leq 12\|a\|_\infty\Big(\|\operatorname{Im} a\|_\infty + 1\Big)\omega(13^{N+1} n)$$

for all sufficiently large n.

Proof. Let q_n be as in the proof of Theorem 3.4 and define b_n by (24) and (25). Then $T(\sigma_n a) = T(\sigma_n b_n)$ and hence

$$\|T^{-1}(\sigma_n a)\| = \|T^{-1}(\sigma_n b_n)\| < 2\Big/\mathrm{dist}\,\Big(0, \mathrm{conv}\,\mathcal{R}(\sigma_n, b_n)\Big)$$

by Proposition 1.3. Since σ_n is a nonnegative apprioximate identity, the inclusion

$$\mathrm{conv}\,\mathcal{R}(\sigma_n b_n) \subset \mathrm{conv}\,\mathcal{R}(b_n)$$

holds. Thus,

$$\|T^{-1}(\sigma_n a)\| < 2\Big/\mathrm{dist}\,\Big(0, \mathrm{conv}\,\mathcal{R}(b_n)\Big) =: 2/D_n.$$

The assertion now follows as in the proof of Theorem 3.4. ■

We remark that this theorem extends to quarter-plane Toeplitz operators and their higher-dimensional versions and that it also has analogues for Wiener-Hopf integral operators.

Appendix

Suppose $a \in L^\infty \setminus \{0\}$ and $\mathrm{conv}\,\mathcal{R}(a) \subset L$ where L is a straight line passing through the origin. On replacing a by γa with an appropriate $\gamma \in \mathbf{T}$, we can without loss of generality assume that a is real-valued. Put

$$m = \mathrm{ess\,inf}\, a, \quad M = \mathrm{ess\,sup}\, a.$$

If $0 \le m$ or $M \le 0$, then Proposition 1.1 tells us that $T_n(a)$ is invertible for all $n \ge 1$.

Example A.1. Fix $\alpha \in (-\pi, \pi)$ and define a by

$$a(e^{i\theta}) = 2(\cos\theta + \cos\alpha).$$

It is easily seen that

$$\det T_n(a) = \det \begin{pmatrix} 2\cos\alpha & 1 & 0 & \dots & 0 \\ 1 & 2\cos\alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2\cos\alpha \end{pmatrix} = \frac{\sin(n+1)\alpha}{\sin\alpha}.$$

Hence, if α/π is irrational then $m < 0 < M$ and $\det T_n(a) \ne 0$ for all $n \ge 1$. If α/π is rational and $|\cos\alpha| < 1$, then again $m < 0 < M$ but there are infinitely many n for which $\det T_n(a) = 0$ and infinitely many n such that $\det T_n(a) \ne 0$. ■

Theorem A.2. *If $a \in L^\infty \setminus \{0\}$ is real-valued then $T_n(a)$ is invertible for infinitely many $n \ge 1$.*

The proof is based on two lemmas.

Lemma A.3. *If $a \in L^\infty \backslash \{0\}$ is real-valued and $\det T_n(a) = 0$ for all $n \geq n_0$ then $a = s/|q|^2$ where*

$$s(t) = \sum_{\nu=-l}^{l} c_l t^l \quad (t \in \mathbf{T})$$

is a real-valued trigonometric polynomial such that $c_l = \overline{c}_{-l} \neq 0$ and

$$q(t) = \prod_{j=1}^{k} (t - \alpha_j) \quad (t \in \mathbf{T})$$

with $0 < |\alpha_j| < 1$ and $k \geq l \geq 0$.

Proof. Let H^∞ denote the functions in L^∞ whose Fourier coefficients with negative indices vanish and let R stand for the rational functions in L^∞.

Assume $\det T_n(a) = 0$ for all $n \geq n_0$. Heinig [9, Satz 6.1 and Lemma 6.1] showed that then a or $\overline{a}$ belongs to $R + H^\infty$. Since a is real-valued, we may suppose that $a \in R + H^\infty$. Hence, there are analytic polynomials p, r and a function $h \in H^\infty$ such that $a = p/r + h$. Write

$$r(t) = t^k r_-(t)\, r_+(t), \quad r_-(t) := \prod_{j=1}^{k} \left(1 - \frac{\alpha_j}{t}\right), \quad r_+(t) := \prod_{j=1}^{m} (t - \beta_j)$$

where $0 < |\alpha_j| < 1$ and $|\beta_j| > 1$. It follows that

$$a(t)|r_-(t)|^2 = t^{-k}\Big(\frac{p(t)\overline{r_-(t)}}{r_+(t)} + h(t)\big(t^k r_-(t)\big)\overline{r_-(t)}\Big)$$

and the function on the right is of the form $t^{-k} g(t)$ with $g \in H^\infty$. Since $a(t)|r_-(t)|^2$ is real-valued, so also is $t^{-k} g(t)$. This implies that

$$t^{-k} g(t) = \sum_{\nu=-l}^{l} c_\nu t^l, \quad l \leq k, \quad c_l = \overline{c}_{-l} \neq 0.$$

Because $|r_-(t)| = \Pi |t - \alpha_j|$, we arrive at the assertion. ■

Lemma A.4. *For $n \in \mathbf{N}$, let*

$$p(n) = \sum_{k=1}^{m} \alpha_k e^{ix_k n}$$

with complex numbers α_k and distinct real numbers $x_k \in [0, 2\pi)$. If $p(n) \to 0$ as $n \to \infty$, then $\alpha_1 = \ldots = \alpha_m = 0$.

Proof. Let $l \in \mathbf{N}$ and consider the values of p at the points $l, l+1, \ldots, l+m-1$:

$$\sum_{k=1}^{m} \alpha_k e^{ix_k(l+\nu)} = p(l+\nu) \ \text{ where } \ \nu = 0, 1, \ldots, m-1.$$

We may rewrite these m equations in the form

$$A \operatorname{diag}(e^{ix_k l})_{k=1}^{m}\, \alpha = \mu(l)$$

where $\alpha := \text{column}\,(\alpha_k)_{k=1}^m$, $\mu(l) := \text{column}\,(p(l+\nu))_{\nu=0}^{m-1}$, and A is the invertible Vandermonde matrix

$$A := \begin{pmatrix} 1 & 1 & \dots & 1 \\ e^{ix_1} & e^{ix_2} & \dots & e^{ix_m} \\ \vdots & \vdots & & \vdots \\ e^{i(m-1)x_1} & e^{i(m-1)x_2} & \dots & e^{i(m-1)x_m} \end{pmatrix}.$$

Hence,

$$\alpha = \text{diag}\,(e^{-ix_k l})_{k=1}^m A^{-1}\mu(l).$$

Since the entries of A and thus of A^{-1} are independent of l and the components of $\mu(l)$ go to zero as l approaches infinity, it follows that the components of α must also tend to zero as l increases to infinity. Because α does actually not depend on l, we arrive at the conclusion that $\alpha = 0$. ∎

Proof of Theorem A.2. Suppose $\det T_n(a) = 0$ for all $n \geq n_0$. Then, by Lemma A.3, a is necessarily a rational function. By a result of Gorodetsky [6] and Trench [16] (also see [1]), we therefore have

$$\det T_n(a) = \sum_{s=0}^{N}\sum_{j=1}^{M_s}(c_{j,s}e^{z_{j,s}n})n^s \tag{A1}$$

for all $n \geq 1$ with certain nonnegative integers N, M_s and complex numbers $c_{j,s}, z_{j,s}$. In (A1), we may assume that $\text{Im} z_{j,s} \in [0, 2\pi)$ and $z_{j_1,s} \neq z_{j_2,s}$ whenever $j_1 \neq j_2$. If $c_{j,s} = 0$ for all j, s, then $\det T_n(a) = 0$ for all $n \geq 1$, which easily implies that all the matrices $T_n(a)$ $(n \geq 1)$ are zero matrices. As the case where a vanishes identically is excluded, we see that not all the numbers $c_{j,s}$ in (A1) are zero.

Let $q := \max \text{Re}\, z_{j,s}$ and rewrite (A1) in the form

$$\det T_n(a) = e^{qn}\sum_{s=0}^{N}\sum_{j=1}^{M_s}(c_{j,s}e^{y_{j,s}n})n^s \tag{A2}$$

with $y_{j,s} := z_{j,s} - q$. Clearly, $\text{Re}\, y_{j,s} \leq 0$ for all j, s. There is a largest $s_0 \in \{0, 1, \dots, N\}$ such that $\text{Re}\, y_{j,s_0} = 0$ and $c_{j,s_0} \neq 0$ for some j. Suppose

$$\text{Re}\, y_{j_1,s_0} = \dots = \text{Re}\, y_{j_m,s_0} = 0$$

and $\text{Re}\, y_{j,s_0} < 0$ for $j \in \{1, \dots, M_{s_0}\} \setminus \{j_1, \dots, j_m\}$. Letting

$$p(n) := \sum_{k=1}^{m} c_{j_k,s_0}e^{y_{j_k,s_0}n} =: \sum_{k=1}^{m} c_{j_k,s_0}e^{ix_k n}$$

we obtain from (A2) that

$$\det T_n(a) = e^{qn}n^{s_0}\Big(p(n) + o(1)\Big) \;\text{ as }\; n \to \infty. \tag{A3}$$

Since $c_{j_k,s_0} \neq 0$ for some j_k, we deduce from Lemma A.4 that there are a $\delta > 0$ and a sequence $\{n_i\}$ of natural numbers such that $n_i \to \infty$ and

$$|p(n_i)| \geq \delta > 0 \;\text{ for all }\; n_i. \tag{A4}$$

Combining (A3) and (A4) we obtain that $\det T_{n_j}(a) \neq 0$ for all sufficiently large n_i. ∎

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