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# The inverse scattering transform for weak Wigner–von Neumann type potentials\*

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## Abstract

In the context of the Cauchy problem for the Korteweg–de Vries equation we extend the inverse scattering transform to initial data that behave at plus infinity like a sum of Wigner–von Neumann type potentials with small coupling constants. Our arguments are based on the theory of Hankel operators.

Keywords: inverse scattering transform, Wigner–von Neumann potential, KdV equation, Hankel operators

Mathematics Subject Classification numbers: 34B20, 37K15, 47B35.

## 1. Introduction

The present paper is concerned with extension of the inverse scattering transform (IST) for the Cauchy problem for the Korteweg–de Vries (KdV) equation

$$\partial_t q - 6q\partial_x q + \partial_x^3 q = 0, \quad (1.1)$$

$$q(x, 0) = q(x), \quad (1.2)$$

with long-range initial profiles  $q$ . In our [1] we have disposed of essentially any decay condition<sup>3</sup> on  $q$  at  $-\infty$  but still required short-range decay at  $+\infty$ . This should not come as a surprise since the KdV is a strongly unidirectional equation (solitons run to the right and radiation waves run to the left) which should translate into different contributions from the behavior

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<sup>3</sup>Which means that (1.1) does not require a boundary condition at  $-\infty$ .

of the data  $q$  at  $\pm\infty$ . In the present paper we put forward a general framework which is robust enough to deal with long range decay at  $+\infty$ . More specifically, we consider decay that is slow enough to include physically relevant Wigner–von Neumann (WvN) resonances (also referred to as spectral singularities) but those resonances are in a way weak (spectral singularities of low order). We note that a WvN resonance could be an embedded eigenvalue as was first shown by WvN in their seminal paper [36]. We refer the interested reader to recent [23, 25] and the extensive literature on embedded spectra cited therein. Note that under our assumptions (see hypothesis 1.1) embedded eigenvalues do not appear and the conditions of hypothesis 1.1 are dictated by what we know about WvN resonances (and long-range potentials as a whole). Our conditions can be relaxed as we fill in the gaps in our understanding of long-range scattering (discussed further in section 4). Of course the spectrum in the long-range situation can be so rich and complicated that it is too long of a shot to understand it to our complete satisfaction.

To fix our notation we give a brief review of the classical IST. Recall (see, e.g. [2, 26, 28]) that conceptually the IST is similar to the Fourier transform and consists of three steps:

**Step 1.** (Direct transform)

$$q(x) \longrightarrow S_q,$$

where  $S_q$  is a new set of variables which turns (1.1) into a simple first order linear ODE for  $S_q(t)$  with the initial condition  $S_q(0) = S_q$ .

**Step 2.** (Time evolution)

$$S_q \longrightarrow S_q(t).$$

**Step 3.** (Inverse transform)

$$S_q(t) \longrightarrow q(x, t).$$

Similar methods have also been developed for many other evolution nonlinear PDEs, which are referred to as completely integrable. Each of steps 1–3 involves solving a linear integral equation that allows us to analyze integrable systems at the level unreachable by neither direct numerical methods nor standard PDE techniques.

This formalism works smoothly in two basic cases (we refer to as classical): when

$$\int_{\mathbb{R}} (1 + |x|) q(x) dx < \infty \quad (\text{the short-range case}) \tag{1.3}$$

and when  $q$  is periodic. We will only be concerned with the former<sup>4</sup> which goes as follows. Associate with  $q$  the full line (self-adjoint) Schrodinger operator  $\mathbb{L}_q = -\partial_x^2 + q(x)$ . For its spectrum  $\sigma(\mathbb{L}_q)$  we have

$$\sigma(\mathbb{L}_q) = \sigma_d(\mathbb{L}_q) \cup \sigma_{ac}(\mathbb{L}_q),$$

where the discrete component  $\sigma_d(\mathbb{L}_q) = \{-\kappa_n^2\}$  is finite and for the absolutely continuous one has  $\sigma_{ac}(\mathbb{L}_q) = [0, \infty)$ . There is no singular continuous spectrum. The Schrödinger equation

$$\mathbb{L}_q \psi = k^2 \psi \tag{1.4}$$

<sup>4</sup>For the latter we refer to [14].

has two (linearly independent) Jost solutions  $\psi_{\pm}(x, k)$ , i.e. solutions satisfying

$$\psi_{\pm}(x, k) = e^{\pm ikx} + o(1), \quad \partial_x \psi_{\pm}(x, k) \mp ik\psi_{\pm}(x, k) = o(1), \quad x \rightarrow \pm\infty. \tag{1.5}$$

Since  $q$  is real,  $\overline{\psi_+}$  also solves (1.4) and one can easily see that the pair  $\{\psi_+, \overline{\psi_+}\}$  forms a fundamental set for (1.4). Hence  $\psi_-$  is a linear combination of  $\{\psi_+, \overline{\psi_+}\}$ . We write this fact as follows ( $k \in \mathbb{R}$ )

$$T(k)\psi_-(x, k) = \overline{\psi_+(x, k)} + R(k)\psi_+(x, k), \quad (\text{basic scattering identity}), \tag{1.6}$$

where  $T$  and  $R$  are called the *transmission* and (right)*reflection* coefficient respectively. The identity (1.6) is totally elementary but serves as a basis for inverse scattering theory. As is well-known (see, e.g. [26]), the triple

$$S_q = \{R, (\kappa_n, c_n)\}, \tag{1.7}$$

where  $c_n = \|\psi_+(\cdot, i\kappa_n)\|^{-1}$ , determines  $q$  uniquely and is called the scattering data for  $\mathbb{L}_q$ . Computing  $S_q$  ends step 1. Note that this set can be constructed for any  $q \in L^1$  with possibly infinitely many  $(\kappa_n, c_n)$ .

Step 2 is computationally easy. Lax pair considerations readily yield

$$S_q(t) = \{R(k) \exp(8ik^3t), \kappa_n, c_n \exp(8\kappa_n^3t)\}. \tag{1.8}$$

Step 3 amounts to solving the inverse scattering problem of recovering the potential  $q(x, t)$  (which now depends on  $t \geq 0$ ) from  $S_q(t)$  via any of the three main methods: the Gelfand–Levitan–Marchenko equation, the trace formula, and the Riemann–Hilbert problem (put in the historic order).

We emphasize that while steps 1–2 remain valid for essentially any real  $q$  supporting Jost solutions  $\psi_{\pm}(x, k)$ , step 3 breaks down in general for long-range potentials (i.e. potentials for which (1.3) does not hold). In fact, the latter occurs already in the case of  $q(x) = O(x^{-2})$ ,  $x \rightarrow \pm\infty$  (i.e. slightly worse than (1.3)). The first example to this effect is explicitly constructed in [3]. The authors present two distinct potentials with such decay and no bound states which share the same reflection coefficient. Thus we have non-uniqueness in solving step 3. The reason for the loss of uniqueness is not explained in [3] but it seems plausible that an approximation of the Jost solution by a suitable Bessel function would capture the singular behavior of  $\psi_{\pm}(x, k)$  near  $k = 0$  (the edge of the a.c. spectrum) which in turn might suggest what pieces of data need to be added to  $S_q$  to resolve the issue<sup>5</sup> (see [4] for some relevant results). For potentials  $q(x) = O(x^{-\alpha})$ ,  $1 < \alpha < 2$ , as is shown in [37], even nice potentials have the Jost solution with an erratic behavior at  $k = 0$ . In some terminology, real points of discontinuity of the Jost solution are referred to as *spectral singularities*. Since the absence of such points is one of the main assumptions in the inverse scattering method, it is reasonable to link non-uniqueness in the case of  $q(x) = O(x^{-\alpha})$ ,  $1 < \alpha \leq 2$ , to a spectral singularity at  $k = 0$ . Note that even though  $k = 0$  is the only possible spectral singularity for a summable potential it is a good open problem [5] to find the extra data that restore uniqueness. If  $q \notin L^1$  then the situation becomes even worse as (1.4) need not have a Jost solution and all steps 1–3 fail in general. Extending the inverse scattering procedure to such potentials is currently out of reach and any essential step towards its solution is important.

Our aim here is to identify a class of  $L^2$  non-integrable potentials supporting a rich set of spectral singularities which nevertheless can be uniquely restored by the classical scattering

<sup>5</sup> We emphasize that  $x^{-2}$  decay may produce infinite (negative) discrete spectrum.

data (1.7). We came across this class while studying its well-known representatives, oscillatory potentials of the form

$$q_\gamma(x) = (A/x) \sin 2\omega x, \quad \gamma := |A/(4\omega)|, \tag{1.9}$$

typically referred to as WvN type potentials. The scattering theory can be developed along the same lines with its short-range counterpart except for the fact the points  $k = \pm\omega$  are spectral singularities of order  $\gamma$  [22]. In the WvN potential community the point  $\omega^2$  is called a *WvN resonance*. In our recent [33] we explicitly construct three potentials with the same (single) resonance  $\omega^2$  and the same set (1.7). Note that  $k = 0$  is not a spectral singularity in those examples meaning that WvN resonances may also cause non-uniqueness. The main goal of our note is to show that for small  $\gamma$  (weak coupling constant) non-uniqueness may only come from a spectral singularity at zero.

Our interest in WvN potentials is inspired in part by the work of Matveev (see [27] and the literature cited therein) and his proposal [12]: ‘a very interesting unsolved problem is to study the large time behavior of the solutions to the KdV equation corresponding to the smooth initial data like  $cx^{-1} \sin 2kx$ ,  $c \in \mathbb{R}$ . Depending on the choice of the constant  $c$  the related Schrödinger operator might have finite or infinite or zero number of the negative eigenvalues. The related inverse scattering problem is not yet solved and the study of the related large times evolution is a very challenging problem’.

In the present paper we deal with initial data (1.2) subject to

**Hypothesis 1.1.** Let  $q$  be a real locally integrable function subject to

- (1) (Decay)  $\mathbb{L}_q$  supports Jost solutions  $\psi_\pm$  such that<sup>6</sup>

$$Y_\pm(x, k) := e^{\mp ikx} \psi_\pm(x, k + i0) - 1 \in H^2.$$

- (2) (Piecewise continuity of the reflection coefficient)  $R$  is piecewise continuous on  $\mathbb{R} \setminus \{\pm\omega_j\}$  and has a jump discontinuity at each  $\pm\omega_j$ ;
- (3) (Jump size)

$$\sup_j \frac{1}{2} |R(\pm\omega_j + 0) - R(\pm\omega_j - 0)| < 1.$$

- (4) ( $L^2$  type decay)  $R \in L^2$ ;
- (5) (Discrete spectrum)  $\sigma_d(\mathbb{L}_q)$  is finite.

The set of such potentials is quite broad but its description in terms of  $q$  alone is out of reach. It can however be describes (or perhaps even characterized) via the spectral measure of the underlying Schrodinger operator  $\mathbb{L}_q$ . Such descriptions though are rarely useful in the IST context as steps 1–3 stated in terms of spectral measure work well only in the setting of periodic potentials and some of their generalizations (see the recent [6] for the spectral approach to almost periodic potentials). For this reason we only outline here how it can be done. The details will be provided elsewhere in a more general case.

Recall that the spectral measure (or rather matrix) of  $\mathbb{L}_q$  can be obtained in terms of the (scalar) spectral measures  $\rho_\pm$  associated with two half-line Schrodinger operators with (say) a Dirichlet boundary condition at (say) zero (see e.g. [34]). In the context of  $L^2$  potentials (our potentials are  $L^2$ ) such measures are completely characterized in the beautiful paper [20]. It is important that three out of four conditions in this characterization follow from the

<sup>6</sup>Here  $H^2$  stands for the Hardy space of the upper half plane (see section 2).

Zakharov–Faddeev trace formula (4.7) below, which is extended in [20] to  $L^2$  potentials, all conditions being very mild. On the other hand, detailed descriptions of the Jost solution, the Weyl  $m$ -function, and the spectral measure<sup>7</sup> of the half-line (Dirichlet) Schrodinger operator with WvN type potentials (and their sums, finite or even infinite) are given in [17, 22] (see section 4). This way we construct two spectral measures  $\rho_{\pm}(E)$  with a desirable behavior for  $E > \varepsilon > 0$ . It remains to prescribe a ‘correct’ behavior of  $\rho_{\pm}(E)$  for  $E < \varepsilon$ . This is not done in [17, 22] but it is where the Killip–Simon characterization [20] comes in handy as it allows us to assign only finitely many pure negative points (eigenvalues) and a desirable behavior at  $E = 0$ , all within  $L^2$  potentials (and without altering  $\rho_{\pm}(E)$  on  $(\varepsilon, \infty)$ ). Consequently, we now have two half-line Weyl  $m$ -functions  $m_{\pm}$  and hence the  $(2 \times 2)$  Weyl matrix  $M$  which representing measure is the spectral measure of  $\mathbb{L}_q$  with all conditions of hypothesis 1.1 satisfied. Then one can find  $q$  on each  $\mathbb{R}_{\pm}$  from  $\rho_{\pm}$  via the inverse spectral method of Gelfand–Levitan–Marchenko [26] or the whole  $q$  via its full line adaptation [5]. As was mentioned above, this however does not yield a convenient description of the conditions in hypothesis 1.1 in terms of  $q$  itself.

**Theorem 1.2 (Main theorem).** *Under hypothesis 1.1 the data  $S_q$  given by (1.7) determine  $q$  uniquely.*

Thus, the principle value of theorem 1.2 is that it takes into account the effect of nonzero resonances (spectral singularities) in the inverse scattering problem. This theorem can be applied to such physically interesting cases as certain potentials supporting finitely many WvN resonances with  $\gamma < 1/2$  or some potentials of the form  $q(x) = p(x)/x$  where  $p(x)$  is a periodic function with a zero mean. However, as we discussed above any explicit description of our class (i.e. in terms of  $q$ ) is out of reach unless we have a suitable description of  $\psi_{\pm}(x, k)$  as  $k \rightarrow 0$ .

We follow standard notation accepted in analysis:  $\mathbb{R}$  is the real line,  $\mathbb{R}_{\pm} = (0, \pm\infty)$ ,  $\mathbb{C}$  is the complex plane,  $\mathbb{C}^{\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$ .  $\bar{z}$  is the complex conjugate of  $z$ . Besides number sets, black board bold letters will also be used for (linear) operators. In particular,  $\mathbb{I}$  denotes the identity operator. We write  $f(x) \sim g(x), x \rightarrow x_0$ , ( $x_0$  may be infinite) if  $\lim(f(x) - g(x)) = 0, x \rightarrow x_0$ .

The paper is organized as follows. In section 2 we give a brief introduction to the theory of Hankel operators and state explicitly what will be used. In section 3 we give a detailed proof of the main theorem. Section 4 is devoted to applications of the main theorem to WvN type potentials. In the final section 5 we apply the main theorem to the analytic factorization problem of the Riemann–Hilbert problem arising in the IST for the KdV equation.

## 2. Hankel operators, basic definitions and important facts

Our approach is based upon techniques of the Hankel operator. Since the Hankel operator is not a conventional tool in inverse problems for the reader’s convenience we give a brief introduction to the theory of Hankel operators and statements of what will be used.

Recall (see e.g. [13]) that a function  $f$  analytic in  $\mathbb{C}^{\pm}$  is in the Hardy space  $H^2(\mathbb{C}^{\pm})$  if

$$\|f\|_2^2 := \sup_{y>0} \int_{\mathbb{R}} |f(x \pm iy)|^2 dx < \infty.$$

We set  $H^2 = H^2(\mathbb{C}^+)$ . It is a fundamental fact that  $f(z) \in H^2(\mathbb{C}^{\pm})$  has non-tangential boundary values  $f(x \pm i0)$  for almost every (a.e.)  $x \in \mathbb{R}$  and  $H^2(\mathbb{C}^{\pm})$  are Hilbert spaces with the

<sup>7</sup>These functions are of course closely related.

inner product induced from  $L^2$ :

$$\langle f, g \rangle_{H^2_{\pm}} = \langle f, g \rangle = \int_{\mathbb{R}} f(x) \bar{g}(x) dx.$$

It is well-known that  $L^2 = H^2(\mathbb{C}^+) \oplus H^2(\mathbb{C}^-)$ , the orthogonal (Riesz) projection  $\mathbb{P}_{\pm}$  onto  $H^2(\mathbb{C}^{\pm})$  being given by

$$(\mathbb{P}_{\pm}f)(x) = \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s) ds}{s - (x \pm i0)}. \tag{2.1}$$

A Hankel operator is an infinitely dimensional analog of a Hankel matrix, a matrix whose  $(j, k)$  entry depends only on  $j + k$ . In the context of integral operators the Hankel operator is usually defined as an integral operator on  $L^2(\mathbb{R}_+)$  whose kernel depends on the sum of the arguments

$$(\mathbb{H}f)(x) = \int_0^{\infty} h(x + y)f(y)dy, \quad f \in L^2(\mathbb{R}_+), \quad x \geq 0, \tag{2.2}$$

and it is this form that Hankel operators typically appear in the inverse scattering formalism. We however consider Hankel operators on  $H^2$  (cf [18, 29]).

Let  $(\mathbb{J}f)(x) = f(-x)$  be the reflection operator in  $L^2$ . It is clearly an isometry with the obvious property

$$\mathbb{J}\mathbb{P}_{\mp} = \mathbb{P}_{\pm}\mathbb{J}. \tag{2.3}$$

**Definition 2.1 (Hankel operator).** Let  $\varphi \in L^{\infty}$ . The operator  $\mathbb{H}(\varphi)$  defined by

$$\mathbb{H}(\varphi)f = \mathbb{J}\mathbb{P}_-(\varphi f), \quad f \in H^2, \tag{2.4}$$

is called the Hankel operator with the symbol  $\varphi$ .

It immediately follows from the definition and (2.3) that  $\|\mathbb{H}(\varphi)\| \leq \|\varphi\|_{\infty}$  and if  $\mathbb{J}\varphi = \bar{\varphi}$  then  $\mathbb{H}(\varphi)$  is selfadjoint on  $H^2$ . A more subtle statement (Hartman’s theorem) says that  $\mathbb{H}(\varphi)$  is compact iff  $\varphi = \varphi_1 + \varphi_2$  where  $\varphi_1$  is a function continuous on the closed real line and  $\varphi_2$  is analytic and uniformly bounded in the upper half plane. However if  $\varphi$  is piecewise continuous with jump discontinuities then  $\mathbb{H}(\varphi)$  is no longer compact. The following deep theorem [31] plays a crucial role in our consideration.

**Theorem 2.2 (Power, 1978).** Let  $\mathbb{J}\varphi = \bar{\varphi}$  (i.e.  $\mathbb{H}(\varphi)$  is selfadjoint),  $\varphi$  decay at  $\pm\infty$ . Suppose that  $\varphi$  is piecewise continuous away from some points  $\{\pm\omega_j\}$  (may be infinitely many) and

$$\alpha_j := \frac{1}{2} |\varphi(\omega_j + 0) - \varphi(\omega_j - 0)|$$

(assuming that the limits exist and are different). Then for the essential spectrum of  $\mathbb{H}(\varphi)$  we have

$$\sigma_{\text{ess}}(\mathbb{H}(\varphi)) = \cup_j [-\alpha_j, \alpha_j].$$

The actual statement of theorem 2.2 given in [31] is more general.

Note that the Hankel operator  $\mathbb{H}$  defined by (2.2) is unitary equivalent to  $\mathbb{H}(\varphi)$  with the symbol  $\varphi$  equal to the Fourier transform of  $h$ . We emphasize though that the form (2.2) does not prove to be convenient for our purposes and also  $h$  is in general not a function but a distribution.

In integral form (2.2) Hankel operators appeared naturally already in the classical papers of Faddeev and Marchenko [26] on inverse scattering. However, the well-developed theory of this class of operators (and even the name itself) was not used at that time. In the KdV context, the Fredholm determinant of  $\mathbb{I} + \mathbb{H}$  appears to be studied for the first time in [30]. In recent [7, 10] some ideas of [30] were extended far beyond the KdV case.

In the conclusion of this section we mention that the formulation of Faddeev–Marchenko inverse scattering theory in terms of Hankel operators defined as in definition 2.1 (adjusted to the unit circle) and the techniques stemming from this theory appeared only in the present century in [15, 35]. However, the inverse scattering problem was studied therein only for Jacobi operators and not in the context of integrable systems.

### 3. Proof of the main theorem

Rewrite (1.6) in the form

$$Ty_- = \bar{y}_+ + R_x y_+, \tag{3.1}$$

where

$$y_{\pm}(x, k) := e^{\mp ikx} \psi_{\pm}(x, k), \quad R_x(k) := e^{2ikx} R(k). \tag{3.2}$$

The function  $y := y_+$  will be used more frequently. Let us regard (3.1) as a Hilbert–Riemann problem of determining  $y_{\pm}$  by given  $T, R$  which we solve by Hankel operator techniques. The potential  $q$  can then be easily found from either  $y_{\pm}$ .

As is well-known, for real  $k$  we have

$$T(-k) = \overline{T(k)}, \quad R(-k) = \overline{R(k)}, \quad |T(k)|^2 + |R(k)|^2 = 1, \tag{3.3}$$

and

$$T(k) = \prod_n \frac{k + i\kappa_n}{k - i\kappa_n} \exp \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(1 - |R(s)|^2) ds}{s - k} \right].$$

Due to conditions 1, 4, 5 of hypothesis 1.1 the function  $Ty_-$  in (3.1) is meromorphic in  $\text{Im } k > 0$  with finitely many simple poles at  $i\kappa_n$ , and  $T(k) \rightarrow 1, k \rightarrow \infty$ , in  $\mathbb{C}^+$ . Compute its residues. It follows from (1.6) that we also have

$$T(k) = \frac{2ik}{W(\psi_-, \psi_+)}, \tag{3.4}$$

where  $W(f, g) := fg' - f'g$  is the Wronskians. Recall that if  $k_0$  is a zero of  $W(\psi_-, \psi_+)$  then  $\psi_+(x, k_0) = \mu_0 \psi_-(x, k_0)$  (linearly dependent) with some  $\mu_0 \neq 0$ , that occurs only for  $k_0 \in i\mathbb{R}_+$  such that  $k_0^2 = -\kappa_0^2$ , where  $-\kappa_0^2$  is a bound state of  $\mathbb{L}_q$ . Next, from the well-known (and easily verifiable) identity

$$\partial_k W(\psi_-(x, k), \psi_+(x, k)) = 2k \int_{\mathbb{R}} \psi_-(s, k) \psi_+(s, k) ds$$

one has

$$\partial_k W(\psi_-(x, k), \psi_+(x, k))|_{k=i\kappa_0} = 2i\kappa_0 \mu_0^{-1} \int_{\mathbb{R}} \psi_+(s, i\kappa_0)^2 ds, \tag{3.5}$$



which means that  $i\kappa_0$  is a simple zero of  $W(\psi_-, \psi_+)$ . It follows from (3.5) and (3.4) that

$$\begin{aligned} \operatorname{Res}_{k=i\kappa_0} T &= \frac{2ik}{\partial_k W(\psi_-, \psi_+)} \Big|_{k=i\kappa_0} = i\mu_0 \left( \int_{\mathbb{R}} \psi_+(s, i\kappa_0)^2 ds \right)^{-1} \\ &= i\mu_0 \|\psi_+(\cdot, i\kappa_0)\|_2^{-2} = i\mu_0 c_0, \end{aligned}$$

where  $c_0$  is the (right) norming constant of the bound state  $-\kappa_0^2$ . Therefore,

$$\begin{aligned} \operatorname{Res}_{k=i\kappa_n} T(k)y_-(x, k) &= y_-(x, i\kappa_n) \operatorname{Res}_{k=i\kappa_n} T \\ &= i\mu_n y_-(x, i\kappa_n) c_n = ic_n e^{-2\kappa_n x} y(x, i\kappa_n), \end{aligned}$$

and taking condition 1 of hypothesis 1.1 into account one has that for each fixed  $x$

$$T(k)y_-(x, k) - 1 - \sum_n \frac{ic_{x,n}}{k - i\kappa_n} y(x, i\kappa_n) \in H^2, \tag{3.6}$$

where  $c_{x,n} := c_n e^{-2\kappa_n x}$ . Rewrite now (3.1) in the form

$$\begin{aligned} T(k)y_-(x, k) - 1 - \sum_n \frac{ic_{x,n}}{k - i\kappa_n} y(x, i\kappa_n) &= \overline{(y(x, k) - 1)} + R_x(k)(y(x, k) - 1) \\ &\quad + R_x(k) - \sum_n \frac{ic_{x,n}}{k - i\kappa_n} y(x, i\kappa_n). \end{aligned} \tag{3.7}$$

Due to (3.6), the left-hand side of (3.7). Noticing that the last term of the right-hand side of (3.7) is in  $H^2(\mathbb{C}^-)$ , the application of the Riesz projection  $\mathbb{P}_-$  to (3.7) yields

$$\mathbb{P}_-(\overline{Y} + R_x Y) + \mathbb{P}_- R_x - \sum_n ic_{x,n} \frac{Y(x, i\kappa_n)}{k - i\kappa_n} - \sum_n \frac{ic_{x,n}}{k - i\kappa_n} = 0, \tag{3.8}$$

where  $Y(x, k) := y(x, k) - 1$ . Thus the left Jost solution  $\psi_-$  is gone from the picture as expected. By condition 1 of hypothesis 1.1,  $Y \in H^2$  for any  $x \in \mathbb{R}$ . Since  $\overline{Y} = \mathbb{J}Y$  and by (2.3) we have

$$\mathbb{P}_- \overline{Y} = \mathbb{P}_- \mathbb{J}Y = \mathbb{J} \mathbb{P}_+ Y = \mathbb{J}Y. \tag{3.9}$$

Observing that for any  $f \in H^2$

$$\mathbb{P}_- \frac{f(\cdot)}{\cdot - i\kappa} = \mathbb{P}_- \frac{f(\cdot) - f(i\kappa)}{\cdot - i\kappa} + \mathbb{P}_- \frac{f(i\kappa)}{\cdot - i\kappa} = \mathbb{P}_- \frac{f(i\kappa)}{\cdot - i\kappa}, \tag{3.10}$$

we have by (3.10) that

$$\sum_n ic_{x,n} \frac{Y(x, i\kappa_n)}{\cdot - i\kappa_n} = \mathbb{P}_- \sum_n ic_{x,n} \frac{Y(x, \cdot)}{\cdot - i\kappa_n}. \tag{3.11}$$

Inserting (3.9) and (3.11) into (3.8), we obtain

$$\mathbb{J}Y + \mathbb{P}_- \left( R_x - \sum_n \frac{ic_{x,n}}{\cdot - i\kappa_n} \right) Y = -\mathbb{P}_- \left( R_x - \sum_n \frac{ic_{x,n}}{\cdot - i\kappa_n} \right).$$

Applying  $\mathbb{J}$  to both sides of this equation yields

$$(\mathbb{I} + \mathbb{H}(\varphi))Y = -\mathbb{H}(\varphi)1, \tag{3.12}$$

where  $\mathbb{H}(\varphi)$  is the Hankel operator defined in definition 2.1 with symbol

$$\varphi(k) = \varphi_x(k) = R_x(k) - \sum_n \frac{ic_{x,n}}{k - i\kappa_n}. \tag{3.13}$$

Due to (3.3),  $\mathbb{J}\varphi = \bar{\varphi}$  and hence  $\mathbb{H}(\varphi)$  is selfadjoint. Note that  $\mathbb{H}(\varphi)1$  on the right-hand side of (3.12) should be interpreted as

$$\mathbb{H}(\varphi)1 = \mathbb{P}_+\bar{\varphi} \in H^2,$$

due to condition 4 of hypothesis 1.1.

We show that  $\mathbb{I} + \mathbb{H}(\varphi)$  is positive definite and hence (3.12) is uniquely solvable for  $Y(x, k)$  for any real  $x$ .

We show first that  $\mathbb{H}(\varphi)$  with

$$\varphi(k) = \frac{-ic}{k - i\kappa}$$

is semi-positive definite for any positive  $c, \kappa$ . To this end, for  $f \in H^2$  consider the quadratic form  $\langle \mathbb{H}(\varphi)f, f \rangle$ . By (3.10) we have

$$\begin{aligned} \langle \mathbb{H}(\varphi)f, f \rangle &= \langle \mathbb{J}\mathbb{P}_-\varphi f, f \rangle \\ &= f(i\kappa) \langle \mathbb{J} \frac{-ic}{\cdot - i\kappa}, f \rangle = icf(i\kappa) \langle \frac{1}{\cdot + i\kappa}, f \rangle \\ &= icf(i\kappa) \overline{\langle f, \frac{1}{\cdot + i\kappa} \rangle} = 2\pi c |f(i\kappa)|^2 \geq 0. \end{aligned}$$

Here at the last step we used the Cauchy formula. We can now conclude that the operator

$$\mathbb{H} \left( \sum_n \frac{-ic_{x,n}}{\cdot - i\kappa_n} \right) = \sum_n \mathbb{H} \left( \frac{-ic_{x,n}}{\cdot - i\kappa_n} \right)$$

is semi-positive definite for any  $x$ .

Thus, the problem now boils down to showing that  $\mathbb{I} + \mathbb{H}(R_x)$  is positive definite for any  $x$ . It follows from conditions 2 and 3 of hypothesis 1.1 that

$$\begin{aligned} \sup_j \frac{1}{2} |R_x(\omega_j + 0) - R_x(\omega_j - 0)| &= \sup_j \frac{1}{2} |e^{2i\omega_j x} (R(\omega_j + 0) - R(\omega_j - 0))| \\ &= \sup_j \frac{1}{2} |R(\omega_j + 0) - R(\omega_j - 0)| < 1 \end{aligned}$$

and by theorem 2.2

$$\sigma_{\text{ess}}(\mathbb{H}(R_x)) = \cup_j [-\alpha_j, \alpha_j],$$

where

$$\alpha_j = \frac{1}{2} |R(\omega_j + 0) - R(\omega_j - 0)|.$$

One now concludes that  $\lambda = \pm 1 \notin \sigma_{\text{ess}}(\mathbb{H}(R_x))$  and therefore  $\lambda = -1$  could only be an eigenvalue of finite multiplicity. It remains to show that it is not the case. We proceed by contradiction assume  $\lambda = -1$  is an eigenvalue of finite multiplicity and let  $f \in H^2$  be the associated normalized eigenfunction. We then have by (2.3)

$$\langle \mathbb{H}(R_x)f, f \rangle = \langle R_x f, \mathbb{P}_- \mathbb{J} f \rangle = \langle R_x f, \mathbb{J} f \rangle$$

and hence by the Cauchy inequality

$$\begin{aligned} |\langle \mathbb{H}(R_x)f, f \rangle|^2 &\leq \int_{\mathbb{R}} |R(k)|^2 |f(k)|^2 dk \|f\|_2^2 \\ &= \int_{\mathbb{R}} |R(k)|^2 |f(k)|^2 dk \\ &= \int_S |R(k)|^2 |f(k)|^2 dk + \int_{\mathbb{R} \setminus S} |R(k)|^2 |f(k)|^2 dk < \|f\|_2^2 = 1, \end{aligned} \tag{3.14}$$

where  $S$  is a set of positive Lebesgue measure where  $|R| < 1$  a.e. Here we have used the fact that  $f \in H^2$  and hence cannot vanish on  $S$ . The inequality (3.14) implies that  $|\lambda| < 1$  which is a contradiction.

#### 4. Applications to WvN type potentials

Theorem 1.2 applies to a variety of oscillatory potentials decaying as  $O(1/x)$  but without understanding the zero energy behavior of scattering data results could only be partial. To describe them we review some results on WvN type potentials following [17, 22] and adjust them to our setting.

Consider a continuous potential of the form

$$q(x) = q_\gamma(x) + O(x^{-2}), \quad x \rightarrow \pm\infty, \tag{4.1}$$

where  $q_\gamma(x)$  is given by (1.9). Clearly  $q$  is square integrable (but not even integrable) and hence the classical short-range techniques do not apply. Nevertheless this potential supports two Jost solutions<sup>8</sup>  $\psi_\pm(x, k)$  analytic for  $\text{Im } k > 0$  and continuous up to the real line except for  $k = \pm\omega$  (and possibly 0) where they blow up to the order of  $\gamma$ :

$$\psi_\pm(x, k) \sim \frac{u_\pm(x)}{(k - \omega)^\gamma}, \quad k \rightarrow \omega, \text{ Im } k \geq 0. \tag{4.2}$$

Due to the symmetry the same behavior takes place of course at  $-\omega$ . In (4.2) the branch cut is taken along  $\mathbb{R}_-$  and  $u_\pm(x)$  are solutions to  $\mathbb{L}_q u = \omega^2 u$  determined by the asymptotics

$$u_\pm(x) \sim c_\pm x^{-\gamma} \cos \omega x, \quad x \rightarrow \pm\infty, \tag{4.3}$$

with some (complex) constants  $c_\pm$ . Thus  $\psi_\pm(x, k)$  blow up to the order  $\gamma$  at  $k = \pm\omega$  (and possibly also at  $k = 0$ ) but are continuous elsewhere. For large  $|k|$  the behavior is the same as that of the short-range case  $\psi_\pm(x, k) = e^{\pm ik}(1 + O(1/k))$ ,  $\text{Im } k \geq 0$ . Note that in the short-range case

<sup>8</sup> It is been known since the 1950s if not earlier.

$\psi_{\pm}(x, k)$  are continuous on the entire real line. The points  $\pm\omega$  are therefore can be called spectral singularities. In the literature however the energy  $\omega^2$  is referred to as a WvN resonance. Generically,  $u_{\pm}(x)$  are linearly independent but for specially chosen  $q$ 's they may become linearly dependent meaning that the Schrodinger equation  $\mathbb{L}_q u = \omega^2 u$  has then a solution which is square integrable if  $\gamma > 1/2$  and decaying (but not  $L^2$ ) if  $\gamma \leq 1/2$  (this will be our case). In the former  $\omega^2$  is a bound state of  $\mathbb{L}_q$  embedded into the a.c. spectrum and in the latter  $\omega^2$  is called a half-bound state. Note that both bound/half-bound states are very unstable (while a resonance is) and can be destroyed by an arbitrarily small perturbation<sup>9</sup>. Thus we assume that  $W(u_-, u_+) \neq 0$ .

We now consider the transmission coefficients  $T$  and  $R$ . It immediately follows from (4.2) that

$$W(\psi_-, \psi_+) \sim \frac{W(u_-, u_+)}{(k - \omega)^{2\gamma}}, \quad k \rightarrow \omega, \text{ Im } k \geq 0,$$

and by (3.4) we have

$$T(k) = \frac{2i\omega}{W(u_-, u_+)}(k - \omega)^{2\gamma} = O(k - \omega)^{2\gamma}, \quad k \rightarrow \omega, \text{ Im } k \geq 0. \tag{4.4}$$

Thus  $T(k)$  vanishes to the order of  $2\gamma$  as  $k \rightarrow \pm\omega$ . Recall that in the short-range case  $T(k)$  vanishes only at zero. Turn now to  $R(k)$ . For some real neighborhood of  $\omega$  (we again use [17, 22])

$$\overline{\psi_+(x, k)}/\psi_+(x, k) \sim \text{Sgn } A e^{-i\pi\gamma \text{Sgn}(k-\omega)}, \quad k \rightarrow \omega. \tag{4.5}$$

Dividing (1.6) by  $\psi_+$  and taking into account (4.5) yields

$$\begin{aligned} T(k) \psi_-(x, k) / \psi_+(x, k) &= \overline{\psi_+(x, k)}/\psi_+(x, k) + R(k) \\ &\sim \text{Sgn } A e^{-i\pi\gamma \text{Sgn}(k-\omega)} + R(k). \end{aligned}$$

On the other hand, by (4.2) and (4.4) one concludes that

$$R(k) + \text{Sgn } A e^{-i\pi\gamma \text{Sgn}(k-\omega)} = O(k - \omega)^{2\gamma}, \quad k \rightarrow \omega,$$

and hence

$$\lim_{k \rightarrow \omega} e^{i\pi\gamma \text{Sgn}(k-\omega)} R(k) = -\text{Sgn } A. \tag{4.6}$$

Note that the subtle fact that (4.2) holds as  $k \rightarrow \omega$  also along the real line (not only tangentially from  $\mathbb{C}^+$ ) was crucially used as (1.6) holds in general on the real line only.

Let us discuss now the decay property of  $R(k)$ . To this end, we recall the well-known KdV second conservation law (aka the second Zakharov–Faddeev trace formula or the Shabat–Zakharov sum rule)

$$\frac{2\pi}{3} \sum_n \kappa_n^3 + \int_{\mathbb{R}} k^2 \log(1 - |R(k)|^2)^{-1} dk = \frac{\pi}{8} \int_{\mathbb{R}} q(x)^2 dx. \tag{4.7}$$

<sup>9</sup>The situation is reminiscent of generic vs exceptional alternative in the short-range case when the transition  $T(k)$  is generically vanishes linearly a  $k \rightarrow 0$  but exceptional  $q$ 's support nontrivial transition at zero momentum. This situation is extremely unstable and there is no way to tell generic and exceptional potentials apart. On the bright side, this phenomenon almost never matters.

Since  $q \in L^2$ , (4.7) immediately implies that  $R(k)$  must be at least square integrable (indeed  $\log(1 - |R|^2)^{-1} \geq |R|^2$ ), which is the same rate of decay as in the short-range case. Incidentally, due to (3.3) this means that  $T(k) = 1 + O(1/k)$  as  $k \rightarrow \infty$  which is in agreement with the short-range scattering.

Thus, the reflection coefficient  $R$  is continuous away from  $\pm\omega$  (and possibly 0), has at  $\pm\omega$  jump discontinuity of size  $2|\sin \pi\gamma|$ , and decays at  $o(1/k)$ . Recalling the properties of  $R$  in the short-range case we see that the appearance of discontinuities seems to be the only difference. However this phenomenon alone makes the machinery of the inverse scattering break down in a very serious way. As was already mentioned in the introduction, it remains a good open problem.

Yet another circumstance is a poor understanding of the zero energy behavior (also stated in [22] as an open problem). It is shown in [21] that  $q_\gamma$  (a pure NvW type potential) has finite negative spectrum if  $\gamma < \sqrt{1/2}$  but if  $\gamma \geq \sqrt{1/2}$  the negative spectrum (necessarily discrete) is infinite, accumulating to zero. Recall that in the short-range inverse scattering every (negative) bound state requires a norming constant. Since those norming constants  $c_n$  in the KdV context determine locations of solitons (which can be arbitrary) the hope that  $c_n$  may have a pattern of behavior as  $n \rightarrow \infty$  is not justified in general<sup>10</sup>. We believe that there is no hope in trying to adapt the classical inverse scattering to the setting of infinite negative spectrum in general and instead a totally different approach is required, which would also handle the rough behavior of the Jost function at zero (work in progress). In this contribution we are focus on the effect of positive resonances on the inverse scattering, which is already very important, and address zero resonance and infinite negative spectrum elsewhere.

Thus we assume that our  $q$  is such that  $k = 0$  is a regular point of the spectrum of  $\mathbb{L}_q$ . In other words,  $\psi_\pm(x, k)$  are continuous<sup>11</sup> as  $k \rightarrow \omega$  in  $\text{Im } k \geq 0$ . Since the (necessarily imaginary) zeros of the Jost solution interlace with  $i\kappa_n$ , this assumption rules out negative infinite spectrum.

Thus one can now see that if  $\gamma < 1/2$  then all conditions of hypothesis 1.1 are satisfied and we arrive at

**Theorem 4.1.** *Let a continuous  $q(x)$  be of the form (4.1) with some real  $A$  and positive  $\omega$  such that  $\gamma = |A/(4\omega)| \in (0, 1/2)$ . Suppose that (a)  $\omega^2$  is not a half bound state of  $\mathbb{L}_q$  and (b)  $0$  is a regular point of  $\sigma(\mathbb{L}_q)$ . Then the set  $S_q = \{R, (\kappa_n, c_n)\}$  determines  $q$  uniquely from the relation*

$$Y(x, \cdot) = -(\mathbb{I} + \mathbb{H}(\varphi_x))^{-1} \mathbb{H}(\varphi_x) 1. \tag{4.8}$$

Furthermore,  $\omega > 0$  is found from the equation  $|R(\omega)| = 1$ ;  $\gamma$  and the sign of  $A$  is determined from (4.6).

We did not specify how one can obtain  $q$  from (4.8). The obvious one is to find  $q$  directly from (1.4). The most common one appears to be by

$$q(x) = -\partial_x \lim 2ik(\mathbb{I} + \mathbb{H}(\varphi_x))^{-1} \mathbb{H}(\varphi_x) 1, \quad k \rightarrow \infty, \text{Im } k \geq 0. \tag{4.9}$$

The most general one is

$$q(x) = \lim 2k^2 (G(x, k^2) - 1), \quad k^2 \rightarrow \infty, \text{Im } k^2 \geq 0, \tag{4.10}$$

<sup>10</sup> Besides, the results of [37] suggest that the behavior of the Jost solution at zero could be quite messy.

<sup>11</sup> In fact, boundedness of  $|k|^{1/2-\varepsilon} \psi_\pm(x, k)$  at  $k = 0$  for some  $\varepsilon > 0$  would be sufficient.

where

$$G(x, k^2) = \frac{\psi_-(x, k)\psi_+(x, k)}{W(\psi_+(x, k), \psi_-(x, k))}$$

is the diagonal Green’s function. Note that (4.10) is not sensitive to conditions on  $q$  once we replace  $\psi_{\pm}$  with Weyl solutions.

**Corollary 4.2.** *Under the conditions of theorem 4.1, the problem (1.1) and (1.2) has a unique solution given by (4.9) where  $\varphi_x$  is replaced with*

$$\varphi_{x,t}(k) = R(k)e^{8ik^3t+2ikx} - \sum_n \frac{ic_n}{k - i\kappa_n} e^{8\kappa_n^3t - 2\kappa_n x}. \tag{4.11}$$

**Proof.** By the Zakharov–Shabat dressing method [28], multiplying scattering data by  $e^{8ik^3t}$  implies the time evolution (the KdV flow)  $q(x) \rightarrow q(x, t)$ . Thus  $q(x, t)$  given by (4.9) with  $\varphi_x$  replaced with  $\varphi_{x,t}$  solves (1.1) with initial data  $q \in L^2$ . Therefore, by the Bourgain theorem [8]  $q(x, t)$  is the solution to (1.1) and (1.2).  $\square$

The set of potentials in theorem 4.1 is not empty but its description in terms of potentials is likely impossible. Constructing specific potentials can be done as follows. We take the spectral measure of the free ( $q = 0$ ) half-line Schrodinger operator with a Dirichlet boundary condition at zero and perturb it in a small neighborhood of  $\omega^2$  to produce a singularity of order  $\gamma < 1/2$  at  $\omega^2$  of the Weyl  $m$ -function. This procedure gives rise a potential (necessarily  $L^2$  due to [20]) on  $\mathbb{R}_+$  which we continue to  $\mathbb{R}_-$  as an even function. The potential constructed this way will be subject to the conditions of theorem 4.1. We employ a similar idea in the recent [33] to construct a potential,

$$q(x) = \begin{cases} q_0(x), & x \geq 0 \\ q_0(-x), & x < 0 \end{cases},$$

where

$$q_0(x) = -2\partial_x^2 \log(1 + \rho x - (\rho/2) \sin 2x), \quad x \geq 0, \quad \rho > 0,$$

that has two Jost solutions given by

$$\psi_{\pm}(x, k) = \left\{ 1 \pm \left( \frac{e^{\pm ix}}{k+1} - \frac{e^{\mp ix}}{k-1} \right) \frac{\rho \sin x}{1 + \rho|x| - (\rho/2) \sin 2|x|} \right\} e^{\pm ikx}, \quad \pm x \geq 0.$$

Apparently, this potential has the asymptotic behavior (4.1) but  $\psi_{\pm}(x, k)$  are clearly continuous at  $k = 0$  (for each  $x$ ). The potential  $q(x)$  is different at  $\pm\infty$  from the WvN potential  $q_{\gamma}$  with  $\gamma = 2$  by  $O(x^{-2})$  and its negative spectrum has only one bound state. A different construction for  $\gamma = 1/2$  is given in [19] based on a subtle limiting procedure in the Gelfand–Levitan–Marchenko equation. This procedure also yields potentials having a smooth spectral measure at zero and decaying like  $q_{\gamma}$  with  $\gamma = 1/2$ . In fact, more than one resonance point is allowed but the techniques do not yield a description of this class in terms of  $q$ ’s either. Since both constructions produce strong WvN potentials (i.e.  $\gamma \geq 1/2$ ) the reader should be convinced that weak WvNs are not any worse. We however have now a totally different approach (work in progress) based on analyzing the effect of altering the spectral measure on

a small interval  $(-\varepsilon, \varepsilon)$  to remove infinitely many (small) negative bound states and smoothen its behavior at the edge of the a.c. spectrum. We conjecture that such an alteration results in a  $O(x^{-2})$  perturbation for a broad class of potentials.

Theorem 4.1 can be extended to a finite sum of potentials of type (4.1) with a new interesting feature. The set of WvN resonances is merely  $\{\omega_j^2\}$  and each resonance produces a jump<sup>12</sup> of  $R$  at  $\pm\omega_j$  of size  $|2 \sin \pi\gamma_j|$ . This puts us in the setting of the recent deep paper [32], where the spectral and scattering theory for self-adjoint Hankel operators  $\mathbb{H}(\varphi)$  with piecewise continuous  $\varphi$  is developed. Under a mild extra condition, each jump of  $\varphi$  gives rise to an interval of the a.c. spectrum of  $\mathbb{H}(\varphi)$  (cf theorem 2.2) the authors construct *wave operators* realizing unitary equivalence of  $\mathbb{H}(\varphi)$  and the orthogonal sum of simple model Hankel operators responsible for each jump similarly to the famous Faddeev’s solution of the *three particle quantum problem*. In our case it means that the a.c. part of  $\mathbb{H}(\varphi_{x,t})$  is unitary equivalent to the sum of orthogonal Hankel operators and each WvN resonance produces a term in the KdV solution. This situation is very similarly to the nonlinear superposition of solitons and one should expect an analog of nonlinear superposition for radiation waves (i.e. propagating to  $-\infty$ ) with phase velocities  $12\omega_j^2$  (similarly to how each negative bound state  $-\kappa_n^2$  produces a soliton with velocity  $4\kappa_n^2$ ). It is worth noticing that a similar phenomenon is studied in [27] in the context of singular (i.e. with a local double pole singularity) WvN type potential commonly referred to as positons. Note that positons do not interact.

Similarly, theorem 4.1 can be extended along the lines to a more general and physically relevant case of a decaying ‘periodic’ structure modelled by

$$q(x) = p(x)/x,$$

where  $p(x)$  is a small enough periodic function with zero-average. The spectral theoretical basis for this is developed in [17, 22] where  $p$  is expanded into the Fourier series producing an infinite sum of WvN potentials with Fourier frequencies  $\omega_j$  and  $(\gamma_j) \in l^1$ , which guarantees convergences. The KdV flow promises some fascinating dynamics.

If  $\gamma = 1/2$  then the essential spectrum of  $\mathbb{H}(R)$  fills  $[-1,1]$  and  $\mathbb{I} + \mathbb{H}(R)$  is no longer boundedly invertible creating a serious problem to our method. It is interesting to note that this problem occurs only if  $\gamma = 1/2 + n$  for any natural  $n$  but  $\psi$ , which still shows the behavior (4.2), is no longer in  $H^2$  and our approach needs serious modifications. We believe that instead of invertibility of  $\mathbb{I} + \mathbb{H}(R_x)$  we need to study invertibility of  $\mathbb{T}(T/\overline{T}) + \mathbb{H}(R_x)$  where  $\mathbb{T}(\varphi)$  is the Toeplitz operator with symbol  $\varphi$ .

In the conclusion of this section we mention that condition (a) of theorem 4.1 is actually unnecessary but the arguments become more complicated. Note that essentially any compactly supported perturbation turns a half-bound state into a resonance.

### 5. Application to a matrix Riemann–Hilbert problem

In this section we apply theorem 1.2 to  $2 \times 2$  matrix Riemann–Hilbert problem that arises in the Riemann–Hilbert problem approach to the IST.

As is well-known, one can rewrite the time-evolved (1.6) as a meromorphic vector Riemann–Hilbert problem (see e.g. [16]) with the *jump matrix*

$$V = \begin{pmatrix} 1 - |R|^2 & -\overline{R}_{x,t} \\ R_{x,t} & 1 \end{pmatrix}, \tag{5.1}$$

<sup>12</sup> At each  $\omega_j$  the Jost solution exhibits a singularity of type (4.2) but  $\psi$  is still in  $H^2$  as long as  $\gamma_j < 1/2$ .

where as before  $R_{x,t}(k) = \exp(8ik^3t + 2ikx)$  with real  $x, t$ . In the language of the Riemann–Hilbert problem, we can claim that loosely speaking the IST works smoothly iff

$$V = V_- V_+ \tag{5.2}$$

with a unique choice of  $2 \times 2$  matrices  $V_{\pm}$  subject to  $V_{\pm} - I \in H^2(\mathbb{C}^{\pm})$ . Such factorization (5.2) is referred to as a canonical  $L^2$  factorization [9] and is called a matrix Riemann–Hilbert problem in the IST community. Note that if  $1 - |R(k)|^2 > 0$  then  $\text{Re } V$  is positive definite and (5.2) holds [9]. Such situation occurs in the modified KdV case [11] but not in the KdV case as generically  $|R(0)| = 1$  even in the short-range case. In our case  $1 - |R(\omega_j)|^2 = 0$ . However, we still have (5.2). More specifically

**Theorem 5.1.** *Let a reflection coefficient  $R$  in (4.5) be such  $|R(k)| \leq 1$ , and  $|R(k)| < 1$  on a set  $S$  of positive measure. Suppose that  $R$  is piecewise continuous on the closed  $\mathbb{R}$  away from some points  $\{\pm\omega_j\}$  (may be infinitely many). Then the matrix Riemann–Hilbert problem*

$$\begin{aligned} V(k) &= V_-(k) V_+(k), \quad k \in \mathbb{R}, \\ V_{\pm} - I &\in H^2(\mathbb{C}^{\pm}), \end{aligned} \tag{5.3}$$

has a unique solution iff

$$\sup_j \frac{1}{2} |R(\omega_j + 0) - R(\omega_j - 0)| < 1.$$

**Proof.** Our arguments are based on an important result of [24], that being adjusted to our situation, says that the problem (5.3) has unique solution iff  $-1 \notin \sigma_{\text{ess}}(\mathbb{H}(R_{x,t}))$ . But as we have shown in the proof of theorem 1.2 it is always the case under conditions 2 as long as 3 of hypothesis 1.1 is satisfied.  $\square$

In the conclusion of the paper we would like to emphasize that the Riemann–Hilbert problem statement of the IST is extremely powerful tool in the asymptotic analysis of solutions to integrable systems (see [11] in the mKdV case and [16] in the KdV case). However, it does not have that edge in the circle of problems we are concerned with and our Hankel operator approach works well instead.

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## References

- [1] Grudsky S and Rybkin A 2015 Soliton theory and Hankel operators *SIAM J. Math. Anal.* **47** 2283–323
- [2] Ablowitz M and Clarkson P 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (London Mathematical Society Lecture Note Series vol 149) (Cambridge: Cambridge University Press)
- [3] Abraham P B, DeFacio B and Moses H E 1981 Two distinct local potentials with no bound states can have the same scattering operator: a nonuniqueness in inverse spectral transformations *Phys. Rev. Lett.* **46** 1657–9
- [4] Aktosun T, Klaus M and van der Mee C 1993 On the Riemann–Hilbert problem for the one-dimensional Schrödinger equation *J. Math. Phys.* **34** 2651–90
- [5] Aktosun T and Klaus M 2001 Chapter 2.2.4: inverse theory: problem on the line *Scattering* ed E R Pike and P C Sabatier (London: Academic) p 770
- [6] Binder I, Damanik D, Goldstein M and Lukic M 2018 Almost periodicity in time of solutions of the KdV equation *Duke Math. J.* **167** 2633–78
- [7] Blower G and Newsham S L 2021 Tau functions associated with linear systems *Operator Theory, Functional Analysis and Applications* (Operator Theory: Advances and Applications vol 282) ed M A Bastos, L Castro and A Y Karlovich (Basel: Birkhäuser)
- [8] Bourgain J 1993 Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations *Geom. Funct. Anal.* **3** 107–56
- [9] Clancey K F and Gohberg I 1981 *Factorization of Matrix Functions and Singular Integral Operators* (Operator Theory: Advances and Applications vol 3) (Basel: Birkhäuser)
- [10] Doikou A, Malham S J A and Stylianidis I 2021 Grassmannian flows and applications to non-commutative non-local and local integrable systems *Physica D* **415** 132744
- [11] Deift P and Zhou X 1993 A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation *Ann. Math.* **137** 295–368
- [12] Dubard P, Gaillard P, Klein C and Matveev V B 2010 On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation *Eur. Phys. J. Spec. Top.* **185** 247–58
- [13] Garnett J B 2007 *Bounded Analytic Functions* (Graduate Texts in Mathematics vol 236) 1st edn (New York: Springer)
- [14] Gesztesy F and Holden H 2003 *Soliton Equations and their Algebraic-Geometric Solutions Volume I. (1 + 1)-Dimensional Continuous Models* (Cambridge Studies in Advanced Mathematics vol 79) (Cambridge: Cambridge University Press)
- [15] Golinskii L, Kheifets A, Peherstorfer F and Yuditskii P 2011 Scattering theory for CMV matrices: uniqueness, Helson–Szegő and strong Szegő theorems *Integr. Equ. Oper. Theory* **69** 479–508
- [16] Grunert K and Teschl G 2009 Long-time asymptotics for the Korteweg–de Vries equation via nonlinear steepest descent *Math. Phys. Anal. Geom.* **12** 287–324
- [17] Hinton D B, Klaus M and Shaw J K 1991 Embedded half-bound states for potentials of Wigner–von Neumann type *Proc. Lond. Math. Soc.* **62** 607–46
- [18] Nikolski N K 2002 *Operators, Functions, and Systems: An Easy Reading Volume 1: Hardy, Hankel and Toeplitz* (Mathematical Surveys and Monographs vol 92) (Providence, RI: American Mathematical Society) p 461
- [19] Novikov R G and Khenkin G M 1984 Oscillating weakly localized solutions of the Korteweg–de Vries equation *Teor. Mat. Fiz.* **61** 199–213 (Russian)
- [20] Killip R and Simon B 2009 Sum rules and spectral measures of Schrödinger operators with  $L^2$  potentials *Ann. Math.* **170** 739–82
- [21] Klaus M 1982 Some applications of the Birman–Schwinger principle *Helv. Phys. Acta* **55** 49–68
- [22] Klaus M 1991 Asymptotic behavior of Jost functions near resonance points for Wigner–von Neumann type potentials *J. Math. Phys.* **32** 163–74
- [23] Krüger H 2012 On the existence of embedded eigenvalues *J. Math. Anal. Appl.* **395** 776–87

- [24] Litvinchuk G S and Spitkovskii I M 1982 Sharp estimates of the defect numbers of a generalized Riemann boundary value problem, factorization of Hermitian matrix-functions, and some problems on approximations by meromorphic functions *Mat. Sb. (N.S.)* **117** 196–215 (Russian)  
Litvinchuk G S and Spitkovskii I M 1983 *Math. USSR-Sb.* **45** 205–24 (Engl. transl.)
- [25] Lukic M 2013 Schrödinger operators with slowly decaying Wigner–von Neumann type potentials *J. Spectr. Theory* **3** 147–69
- [26] Marchenko V A 1986 *Sturm–Liouville Operators and Applications (Operator Theory: Advances and Applications vol 22)* (Basel: Birkhäuser) pp xii+367 Translated from the Russian by A Iacob
- [27] Matveev V 2002 Positons: slowly decreasing analogues of solitons *Teor. Mat. Fiz.* **131** 44–61 (Russian)  
Matveev V B 2002 *Theor. Math. Phys.* **131** 483–97 (Engl. transl.)
- [28] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons. The Inverse Scattering Method (Contemporary Soviet Mathematics)* (New York: Plenum) Translated from the Russian
- [29] Peller V V 2003 *Hankel Operators and their Applications (Springer Monographs in Mathematics)* (New York: Springer)
- [30] Pöppe C 1984 The Fredholm determinant method for the KdV equations *Physica D* **13** 137–60
- [31] Power S R 1982 *Hankel Operators on Hilbert Space* (Boston: Pitman)
- [32] Pushnitski A and Yafaev D 2015 Spectral and scattering theory of self-adjoint Hankel operators with piecewise continuous symbols *J. Oper. Theory* **74** 417–55
- [33] Rybkin A 2021 The effect of a positive bound state on the KdV solution: a case study *Nonlinearity* **34** 1238–61
- [34] Titchmarsh E C 1962 *Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I* 2nd edn (Oxford: Clarendon)
- [35] Volberg A and Yuditskii P 2002 On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of the intervals or a Cantor set of positive length *Commun. Math. Phys.* **226** 567–605
- [36] von Neumann J and Winger E P 1929 Über merkwürdige diskrete Eigenwerte *Z. Phys.* **30** 465–7
- [37] Yafaev D R 1982 The low energy scattering for slowly decreasing potentials *Commun. Math. Phys.* **85** 177–96