Contemporary Mathematics
Volume **591**, 2013
http://dx.doi.org/10.1090/conm/591/11830

A Note on Muskhelishvili-Vekua Reduction

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Abstract. We focus on classical boundary value problems for the Laplace equation in a plane domain bounded by a nonsmooth curve which has a finite number of singular points. Using a conformal mapping of the unit disk onto the domain, we pull back the problem to the unit disk, which is usually referred to as the Muskhelishvili-Vekua method. The problem in the unit disk reduces to a Toeplitz equation with symbol having discontinuity of second kind. We develop a constructive invertibility theory for Toeplitz operators in the unit disk to derive solvability conditions and explicit formulas for solutions of the boundary value problem.

1. Statement of the problem

Elliptic partial differential equations are known to appear in many applied areas of mathematical physics. To name but a few, we mention mechanics of solid medium, diffraction theory, hydrodynamics, gravity theory, quantum field theory, and many others.

In this paper, we focus on boundary value problems for the Laplace equation in plane domains bounded by nonsmooth curves $\mathcal{C}$. We are primarily interested in domains whose boundaries have a finite number of singular points of oscillating type. By this is meant that the curve may be parametrised in a neighbourhood of a singular point z_0 by $z(r) = z_0 + r \exp(\imath\varphi(r))$ for $r \in (0, r_0]$, where r is the distance of z and z_0 and $\varphi(r)$ is a real-valued function which is bounded while its derivative in general is unbounded at $r = 0$.

There is a vast literature devoted to boundary value problems for elliptic equations in domains with nonsmooth boundary, cf. [**KL91**], [**MNP00**], [**KMR00**] and the references given there. In most of these papers, piecewise smooth curves with corner points or cusps are treated, cf. [**DS00**], [**KKP98**], [**KP03**], [**MS89**], [**Rab99**]. The paper [**RST04**] is of particular importance, for it gives a characterisation of Fredholm boundary value problems in domains with weakly oscillating cuspidal edges on the boundary.

There are significantly fewer works dealing with more complicated curves $\mathcal{C}$. They mostly focus on qualitative properties, such as existence, uniqueness and

2010 *Mathematics Subject Classification.* Primary 30E25; Secondary 45E05, 35Q15.

Key words and phrases. Singular integral equations, nonsmooth curves, boundary value problems.

The authors acknowledge financial support of the DFG (German Research Society), grant TA 289/8-1, and the CONACYT (National Society of Science and Technology), grant 102800.

stability of solutions with respect to small perturbations, see for instance [**Kel66**], [**KM94**]. The present paper deals not only with qualitative investigations of boundary value problems in domains whose boundaries strongly oscillate at singular points but also with constructive solution of such problems.

We restrict ourselves to the Dirichlet problem for the Laplace equation

$$\Delta u := (\partial/\partial x)^2 u + (\partial/\partial y)^2 u = 0 \tag{1.1}$$

in a simply connected domain $\mathcal{D}$ with boundary $\mathcal{C}$ in the plane of variables $(x, y) \in \mathbb{R}^2$. The boundary data are

$$u = u_0 \tag{1.2}$$

on $\mathcal{C}$. Our standing assumption on u_0 is that $u_0 \in L^p(\mathcal{C})$ with some $1 < p < \infty$. We look for a solution u being the real part of a holomorphic function of Hardy-Smirnov class $E^p(\mathcal{D})$ in $\mathcal{D}$. Such functions are known to possess finite nontangential limit values almost everywhere on the boundary, and so equality (1.2) is understood in that sense.

Our setting of the Dirichlet problem follows that of the monograph [**KKP98**] and the results are intimately related to Theorem 1.1 of [**KKP98**, p. 165]. This theorem gives a complete study of the case where the angle at which the tangent of the boundary intersects the real axis has discontinuities of the first kind at angular points of the boundary. Our theory treats also the case where the angle of the tangent has discontinuities of the second kind at angular points, under some assumptions on the oscillations of the tangent. For example, in the case $p = 2$, these assumptions are satisfied if the angle of the tangent has a first kind discontinuity at the angular point. Hence, our treatment is essentially more advanced than that of [**KKP98**]. The approach we apply is different from that of [**KKP98**] and it exploits the modern techniques of Toeplitz operators.

We study the case where $\mathcal{C}$ belongs to the class of so-called sectorial curves. In a forthcoming paper, we shall treat both the Dirichlet and Neumann problems as well as a more general Zaremba problem in domains bounded by spirals including non-rectifiable ones, see [**GT12**].

2. General description of the method

Our approach to the study of elliptic problems in domains with nonsmooth boundary goes back at least as far as [**Mus68**] and [**Vek42**]. It consists in reducing the problem in $\mathcal{D}$ to a singular integral equation on the unit circle by means of a conformal map of the unit disk onto the domain $\mathcal{D}$. The coefficients of the singular integral equation obtained in this way fail in general to be continuous, for they are intimately connected with the derivative of boundary values of the conformal map. This method was successfully used for solving problems in domains with piecewise smooth boundary, where the singular points are corner points or cusps, see [**DS00**], [**KKP98**], [**KP03**]. In this case, the coefficients of the mentioned singular integral equation have discontinuities of the first kind. Since the theory of such equations has been well developed, a sufficiently complete theory of boundary value problems for a number of elliptic equations in domains with piecewise smooth boundary has been constructed. Note that by now the theory of singular integral equations (or, in other terms, the theory of Toeplitz operators) with discontinuities of second kind has been well developed, too. In particular, we use well-known results on Toeplitz operators with sectorial symbols (see for instance [**BS90**] and [**GK92**]) to study problem (1.1), (1.2). Note that boundary value problems in Lipschitz domains for strongly elliptic second order partial differential equations have been studied intensively, see for instance [**Ken94**]. The methods of our paper are not applicable to get new results in the general case. However, the explicit formulas like that of Theorem 5.3 provide an attractive complement to the strong qualitative results of [**Ken94**].

3. Reduction of the Dirichlet problem

The Dirichlet problem is a most frequently encountered elliptic boundary value problem. This is not only because the Dirichlet problem is of great interest in applications in electrostatics, gravity theory, incompressible fluid theory, etc., but also since it is a good model where one tests approaches to other, more complicated, problems.

Let $\mathcal{D}$ be a simply connected bounded domain in the plane of real variables (x, y). The boundary of $\mathcal{D}$ is a closed curve which we denote by $\mathcal{C}$. Consider the Dirichlet problem (1.1), (1.2) in $\mathcal{D}$ with data u_0 on $\mathcal{C}$. As usual, we introduce a complex structure in $\mathbb{R}^2$ by $z = x + \imath y$ and pick a conformal mapping $z = \mathfrak{c}(\zeta)$ of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto the domain $\mathcal{D}$, cf. Riemann mapping theorem. Throughout the paper, we make a standing assumption on the mappings $z = \mathfrak{c}(\zeta)$ under consideration, specifically,

$$\mathfrak{c}'(0) > 0. \tag{3.1}$$

Then problem (1.1), (1.2) can be reformulated as

$$\begin{array}{rcll} \dfrac{1}{|\mathfrak{c}'(\zeta)|^2}\, \Delta U & = & 0 & \text{for} \quad |\zeta| < 1, \\ U & = & U_0 & \text{for} \quad |\zeta| = 1, \end{array} \tag{3.2}$$

where $U(\zeta) := u(\mathfrak{c}(\zeta))$ and $U_0(\zeta) := u_0(\mathfrak{c}(\zeta))$.

For $1 \le p < \infty$, we denote by $H^p(\mathbb{D})$ the Hardy space on the unit disk. By the conformal map $z = \mathfrak{c}(\zeta)$, the space is transported to the so-called Hardy-Smirnov space $E^p(\mathcal{D})$ of functions on $\mathcal{D}$. A holomorphic function f on $\mathcal{D}$ is said to belong to $E^p(\mathcal{D})$ if

$$\sup_{r \in (0,1)} \int_{\mathcal{C}_r} |f(z)|^p \, |dz| < \infty,$$

where $\mathcal{C}_r$ is the push-forward of the circle $|\zeta| = r$ by $z = \mathfrak{c}(\zeta)$. It is easy to see that $f \in E^p(\mathcal{D})$ if and only if

$$\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta)) \in H^p(\mathbb{D}). \tag{3.3}$$

It is then a familiar property of the functions of Hardy class $H^1(\mathbb{D})$ that the function $\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta))$ has finite nontangential limit values almost everywhere on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

If $\mathcal{C}$ is a rectifiable curve, then the function $z = \mathfrak{c}(\zeta)$ is continuous on the closed unit disk $\overline{\mathbb{D}}$, absolutely continuous on the unit circle $\mathbb{T}$ and $(\mathfrak{c}(e^{\imath t}))' = \imath e^{\imath t} \mathfrak{c}'(e^{\imath t})$ almost everywhere on $\mathbb{T}$. It follows from (3.3) that $f(z)$ has finite nontangential limit values almost everywhere on $\mathcal{C}$, and

$$\lim_{r \to 1-} \int_{\mathcal{C}_r} |f(z)|^p \, |dz| = \int_{\mathcal{C}} |f(z)|^p \, |dz|. \tag{3.4}$$

It is well known that for each harmonic function $u(x,y)$ in $\mathcal{D}$ there is an analytic function $f(z)$ in $\mathcal{D}$ whose real part is u. We therefore look for a solution u of problem (1.1) and (1.2), which has the form $u = \Re f$ with $f \in E^p(\mathcal{D})$. There is no restriction of generality in assuming that

$$\Im f(\mathfrak{c}(0)) = 0. \tag{3.5}$$

By the above, we get

$$f(\mathfrak{c}(\zeta)) = \frac{h^+(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}}$$

for $\zeta \in \mathbb{D}$, where h^+ is an analytic function of Hardy class $H^p(\mathbb{D})$.

By Theorem 4 in [**Gol69**, p. 46], the conformal mapping $z = \mathfrak{c}(\zeta)$ is bijective and continuous on the closed unit disk. Hence the function $U(\zeta) = u(\mathfrak{c}(\zeta))$ has finite nontangential limit values almost everywhere on $\mathbb{T}$ and in this way $U(\zeta) = U_0(\zeta)$ is understood on the unit circle $\mathbb{T}$. This enables us to rewrite problem (3.2) in the form

$$\Re\Big(\frac{h^+(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}}\Big) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$, where h^+ is an analytic function of Hardy class $H^p(\mathbb{D})$. This latter problem can in turn be reformulated as

$$\frac{1}{2}\Big(\frac{h^+(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}} + \frac{h^-(\zeta)}{\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}}\Big) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$, where

$$\begin{array}{rcl} h^-(\zeta) & = & \overline{h^+(\zeta)} \\ & = & \overline{h^+\Big(\frac{\zeta}{|\zeta|^2}\Big)} \\ & = & \overline{h^+\Big(\frac{1}{\bar{\zeta}}\Big)} \end{array}$$

can be specified within analytic functions of Hardy class H^p in the complement of the closed unit disk. More precisely,

$$\overline{h^+\Big(\frac{1}{\bar{\zeta}}\Big)}$$

belongs to the Hardy class H^p in the complement of $\bar{\mathbb{D}}$ up to an additive complex constant, if the functions of Hardy class H^p in $\mathbb{C} \setminus \bar{\mathbb{D}}$ are assumed to vanish at infinity. Finally, we transform the Dirichlet problem to

$$a(\zeta)\, h^+(\zeta) + h^-(\zeta) = f(\zeta) \tag{3.6}$$

for $\zeta \in \mathbb{T}$, where

$$a(\zeta) = \Big(\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}\Big)^{1/p} = \exp\Big(-\imath\frac{2}{p}\arg\mathfrak{c}'(\zeta)\Big)$$

and $f(\zeta) = 2U_0(\zeta)\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}$. It is well known from the theory of conformal mappings that

$$\arg\mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg\zeta - \frac{\pi}{2}$$

for $\zeta \in \mathbb{T}$, where $\alpha(\mathfrak{c}(\zeta))$ is the angle at which the tangent of $\mathcal{C}$ at the point $z = \mathfrak{c}(\zeta)$ intersects the real axis. Note that $f \in L^p(\mathbb{T})$.

Let now

$$(S_{\mathbb{T}}f)(\zeta) := \frac{1}{\pi\imath}\int_{\mathbb{T}} \frac{f(\zeta')}{\zeta'-\zeta}\,d\zeta', \quad \zeta \in \mathbb{T},$$

stand for the singular Cauchy integral. If $1 < p < \infty$, then $S_{\mathbb{T}}$ is a bounded operator in $L^p(\mathbb{T})$, and the operators

$$P^{\pm}_{\mathbb{T}} := \frac{1}{2}\,(I \pm S_{\mathbb{T}})$$

prove to be continuous projections in $L^p(\mathbb{T})$ called analytic projections. They are intimately related with the classical decomposition of $L^p(\mathbb{T})$ into the direct sum of traces on $\mathbb{T}$ of Hardy class H^p functions in $\mathbb{D}$ and $\mathbb{C}\setminus\overline{\mathbb{D}}$. Denoting these trace spaces by $H^{p\pm}$ we get

$$P^{\pm}_{\mathbb{T}}\,L^2(\mathbb{T}) = H^{p\pm},$$

whence $P^{\pm}_{\mathbb{T}}\,H^{p\pm} = H^{p\pm}$ and $P^{\pm}_{\mathbb{T}}\,H^{p\mp} = 0$.

Applying $P^{+}_{\mathbb{T}}$ to both sides of equality (3.6) and taking into account that $(P^{+}_{\mathbb{T}}h^{-})(\zeta) = h^{-}(0)$ and $h^{-}(0) = \overline{h^{+}(0)} = h^{+}(0)$, the latter being due to (3.5), we get

$$(T(a)h^{+})(\zeta) + h^{+}(0) = f^{+}(\zeta) \tag{3.7}$$

for $\zeta \in \mathbb{T}$, where $T(a) := P^{+}_{\mathbb{T}}aP^{+}_{\mathbb{T}}$ is a Toeplitz operator with symbol a on $L^p(\mathbb{T})$ and $f^{+}(\zeta) = (P^{+}_{\mathbb{T}}f)\,(\zeta)$ for $\zeta \in \mathbb{T}$. By (3.5) we have to put the additional condition

$$\Im h^{+}(0) = 0. \tag{3.8}$$

We thus arrive at the following result.

THEOREM 3.1.

1) *If $u = \Re f$ with $f \in E^p(\mathcal{D})$ is a solution of the Dirichlet problem in $\mathcal{D}$, then $h^{+}(\zeta) = \sqrt[p]{\mathfrak{c}'(\zeta)}\,f(\mathfrak{c}(\zeta))$ is a solution of equation (3.7).*
2) *If $h^{+} \in H^{p+}$ is a solution of (3.7), then $u(z) = \Re\,(\sqrt[p]{\mathfrak{c}^{-1\prime}(z)}\,h^{+}(\mathfrak{c}^{-1}(z)))$ is a solution of the Dirichlet problem in $\mathcal{D}$.*

PROOF. 1) has already been proved, it remains to show 2). Let $h^{+} \in H^{p+}$ satisfy (3.7). Rewrite this equality in the form $ah^{+} + h^{-} = f$ with h^{-} given by $h^{-} = -P^{-}_{\mathbb{T}}(ah^{+}) + P^{-}_{\mathbb{T}}f$. This latter equality can in turn be rewritten as

$$\frac{1}{2}\Big(\frac{h^{+}(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}} + \frac{h^{-}(\zeta)}{\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}}\Big) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$. Since the function $U_0(\zeta) = u_0(\mathfrak{c}(\zeta))$ is real-valued, it follows that $h^{-}(\zeta) = \overline{h^{+}(\zeta)}$, and so

$$\Re\,\frac{h^{+}(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}} = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$. The function

$$\Re\,\frac{h^{+}(\zeta)}{\sqrt[p]{\mathfrak{c}'(\zeta)}}$$

is harmonic in $\mathbb{D}$ and has nontangential limit values almost everywhere on $\mathbb{T}$ which coincide with $U_0(\zeta)$. Moreover, the function $f(z) := \sqrt[p]{\mathfrak{c}^{-1\prime}(z)}\,h^{+}(\mathfrak{c}^{-1}(z))$ is of Hardy-Smirnov class $E^p(\mathcal{D})$ and $u(x,y) = \Re\,f(z)$ is a solution of the Dirichlet problem in $\mathcal{D}$, as desired. □

COROLLARY 3.2. *If the operator $T(a)$ is invertible on the space H^{p+} and*

$$\begin{array}{rcl} \Im\left(T(a)^{-1}f^{+}\right)(0) & = & 0, \\ \left(T(a)^{-1}1\right)(0) & = & 1, \end{array} \tag{3.9}$$

then the Dirichlet problem in $\mathcal{D}$ has a unique solution of the form

$$u(z)=\Re\Big(\sqrt[p]{\mathfrak{c}^{-1\prime}(z)}\,\Big((T(a)^{-1}f^{+})(\mathfrak{c}^{-1}(z))-\frac{1}{2}(T(a)^{-1}f^{+})(0)\,(T(a)^{-1}1)(\mathfrak{c}^{-1}(z))\Big)\Big),$$

where $f^{+}=P_{\mathbb{T}}^{+}\left(2u_0(\mathfrak{c}(\zeta))\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}\right)$.

PROOF. Applying the operator $T(a)^{-1}$ to (3.7) yields

$$h^{+}(\zeta)+T(a)^{-1}h^{+}(0)=\left(T(a)^{-1}f^{+}\right)(\zeta)$$

for all $\zeta\in\mathbb{T}$. Since both sides of the equality extend to holomorphic functions in the disk, we can set $\zeta=0$, and obtain

$$2\,h^{+}(0)=\left(T(a)^{-1}f^{+}\right)(0)$$

by (3.9). Hence it follows that the solution of (3.7) satisfying (3.8) is unique and it has the form

$$h^{+}(\zeta)=\left(T(a)^{-1}f^{+}\right)(\zeta)-\frac{1}{2}\left(T(a)^{-1}f^{+}\right)(0)(T(a)^{-1}1)(\zeta),$$

as desired. □

REMARK 3.3. Condition (3.9) is actually fulfilled in all cases to be treated in this work.

4. Factorisation of symbols

The results of this section with detailed explanations, proofs and corresponding references can be found in the books [**GK92**], [**LS87**], and [**BS90**].

Let $L^{\infty}(\mathbb{T})$ be the space of all essentially bounded functions on the unit circle $\mathbb{T}$, $H^{\infty\pm}$ the Hardy spaces on $\mathbb{T}$ which consist of the restrictions to $\mathbb{T}$ of bounded analytic functions in $\mathbb{D}$ and $\mathbb{C}\setminus\overline{\mathbb{D}}$, respectively, and $C(\mathbb{T})$ the space of all continuous functions on $\mathbb{T}$.

A bounded linear operator A on a Hilbert space H is said to be normally solvable if its range $\operatorname{im} A$ is closed. A normally solvable operator is called Fredholm if its kernel and cokernel are finite dimensional. In this case, the index of A is introduced as

$$\operatorname{ind} A:=\alpha(A)-\beta(A),$$

where $\alpha(A)=\dim\ker A$ and $\beta(A)=\dim\operatorname{coker} A$.

The symbol $a(\zeta)$ of a Toeplitz operator $T(a)$ is said to admit a p-factorisation if it can be represented in the form

$$a(\zeta)=a^{+}(\zeta)\zeta^{\kappa}a^{-}(\zeta), \tag{4.1}$$

where κ is an integer number,

$$\begin{array}{rclcrcl} a^{+} & \in & H^{q+}, & \quad & a^{-} & \in & H^{p-}, \\ 1/a^{+} & \in & H^{p+}, & \quad & 1/a^{-} & \in & H^{q-}, \end{array} \tag{4.2}$$

p and q are conjugate exponents (i.e., $1/p+1/q=1$), and $(1/a^{+})S_{\mathbb{T}}(1/a^{-})$ is a bounded operator on $L^{p}(\mathbb{T})$.

The functions a_+ and a_- in (4.1) are determined uniquely up to a constant factor. As is proved in [**LS87**], the factorisation is determined uniquely up to a multiplicative constant if it bears properties (4.2) only.

THEOREM 4.1. *An operator $T(a)$ is Fredholm in the space H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation. If $T(a)$ is Fredholm, then* $\operatorname{ind} T(a) = -\kappa$.

THEOREM 4.2. *Let $a \in L^\infty(\mathbb{T})$ and $a(\zeta) \neq 0$ almost everywhere on $\mathbb{T}$. Then at least one of the numbers $\alpha(T(a))$ and $\beta(T(a))$ is equal to zero.*

Combining Theorems 4.1 and 4.2, we get a criterion of invertibility for Toeplitz operators.

COROLLARY 4.3. *An operator $T(a)$ is invertible on H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation with $\kappa = 0$. In this case*

$$(T(a))^{-1} = (1/a^+)P^+_{\mathbb{T}}(1/a^-).$$

PROOF. If $\kappa = 0$ then $\alpha(T(a)) = \beta(T(a))$, and so both $\alpha(T(a))$ and $\beta(T(a))$ vanish. Hence it follows that $T(a)$ is invertible on H^{p+}.

We now establish the formula for the inverse operator $(T(a))^{-1}$. Let $f \in H^{p+}$. Then

$$\begin{aligned}
\big((1/a^+)P^+_{\mathbb{T}}(1/a^-)\big)\,T(a)f &= \big((1/a^+)P^+_{\mathbb{T}}(1/a^-)\big)\,P^+_{\mathbb{T}}(af)\\
&= \big((1/a^+)P^+_{\mathbb{T}}(1/a^-)\big)\,af\\
&= (1/a^+)P^+_{\mathbb{T}}a^+f\\
&= (1/a^+)a^+f\\
&= f,
\end{aligned}$$

and similarly

$$\begin{aligned}
T(a)\,\big((1/a^+)P^+_{\mathbb{T}}(1/a^-)\big)\,f &= P^+_{\mathbb{T}}a\,\big((1/a^+)P^+_{\mathbb{T}}(1/a^-)\big)\,f\\
&= P^+_{\mathbb{T}}a^-P^+_{\mathbb{T}}(1/a^-)f\\
&= P^+_{\mathbb{T}}a^-(1/a^-)f\\
&= f.
\end{aligned}$$

Here we have used the familiar equalities $P^+_{\mathbb{T}}h^-P^+_{\mathbb{T}} = P^+_{\mathbb{T}}h^-$ and $P^+_{\mathbb{T}}h^+P^+_{\mathbb{T}} = h^+P^+_{\mathbb{T}}$ which is valid for all $h^- \in H^{q-} \oplus \{c\}$ and $h^+ \in H^{q+}$. □

Given a nonvanishing function $a \in C(\mathbb{T})$, we denote by $\operatorname{ind}_{a(\mathbb{T})}(0)$ the winding number of the curve $a(\mathbb{T})$ about the origin, or the index of the origin with respect to $a(\mathbb{T})$.

THEOREM 4.4. *Suppose $a \in C(\mathbb{T})$. Then the operator $T(a)$ is Fredholm on the space H^{p+} if and only if $a(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$. Under this condition, the index of $T(a)$ is given by*

$$\operatorname{ind} T(a) = -\operatorname{ind}_{a(\mathbb{T})}(0).$$

We now introduce the concept of sectoriality which is of crucial importance in this paper.

DEFINITION 4.5. A function $a \in L^\infty(\mathbb{T})$ is called p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$ and there is a real number φ_0 such that

$$\sup_{\zeta \in \mathbb{T}} |\arg(\exp(\imath\varphi_0)a(\zeta))| < \frac{\pi}{\max\{p,q\}} \tag{4.3}$$

for all $\zeta \in \mathbb{T}$.

A function $a \in L^\infty(\mathbb{T})$ is said to be locally p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$ and for any $\zeta_0 \in \mathbb{T}$ there is an open arc containing ζ_0, such that (4.3) is satisfied for all ζ in the arc with some $\varphi_0 \in \mathbb{R}$ depending on ζ_0. Each p-sectorial curve is obviously locally p-sectorial.

THEOREM 4.6.

1) *If $a(\zeta)$ is a p-sectorial symbol, then the operator $T(a)$ is invertible in the space H^{p+}.*
2) *If $a(\zeta)$ is a locally p-sectorial symbol, then $T(a)$ is a Fredholm operator in H^{p+}.*

If a symbol a is factorised by a sectorial symbol to a continuous symbol which does not vanish on $\mathbb{T}$, then the operator $T(a)$ is Fredholm.

THEOREM 4.7. *Let $a(\zeta) = c(\zeta)a_0(\zeta)$, where $c \in C(\mathbb{T})$ and $a_0 \in L^\infty(\mathbb{T})$. Then $T(a)$ is Fredholm in H^{p+} if and only if $c(\zeta)$ vanishes at no point of $\mathbb{T}$ and $T(a_0)$ is Fredholm, in which case*

$$\operatorname{ind} T(a) = \operatorname{ind} T(a_0) - \operatorname{ind}_{c(\mathbb{T})}(0).$$

PROOF. This is a straightforward consequence of Theorems 4.1 and 4.4. $\square$

5. Sectorial curves

In this section we consider a simply connected domain $\mathcal{D} \subset\subset \mathbb{R}^2$ whose boundary $\mathcal{C}$ is smooth away from a finite number of points. By this is meant that $\mathcal{C}$ is a Jordan curve of the form

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k,$$

where $\mathcal{C}_k = [z_{k-1}, z_k]$ is an arc with initial point z_{k-1} and endpoint z_k which are located after each other in positive direction on $\mathcal{C}$, and $z_n = z_0$. Moreover, (z_{k-1}, z_k) is smooth for all k.

DEFINITION 5.1. The curve $\mathcal{C}$ is called p-sectorial if, for each $k = 1, \dots, n$, there is a neighbourhood (z_k^-, z_k^+) of z_k on $\mathcal{C}$ and a real number φ_k, such that

$$\sup_{z \in (z_k^-, z_k^+) \setminus \{z_k\}} |\alpha(z) - \varphi_k| < \begin{cases} \dfrac{\pi}{2}, & \text{if } p \ge 2, \\[2ex] \dfrac{\pi}{2}\,(p-1), & \text{if } 1 < p < 2, \end{cases} \tag{5.1}$$

where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at the point z intersects the real axis.

If z_k is a conical point of $\mathcal{C}$, then the angle at which the tangent of $\mathcal{C}$ at z intersects the real axis has jump $j_k < \pi$ when z passes through z_k. It follows that (5.1) is fulfilled at z_k with a suitable φ_k, if $p \ge 2$, and is fulfilled if moreover $j_k < (p-1)\pi$, if $1 < p < 2$. If z_k is a cuspidal point of $\mathcal{C}$, then the angle has jump $j_k = \pi$ when z passes through z_k. Hence, condition (5.1) is violated, i.e., cuspidal points are prohibited for sectorial curves. Yet another example of prohibited behaviour is described by the curve $z(t) = t(1 + \imath \sin(1/t))$ with $|t| < \varepsilon$, which oscillates rapidly near the origin.

THEOREM 5.2. *Suppose $\mathcal{C}$ is p-sectorial for $1 < p < \infty$. Then the Toeplitz operator (3.7) corresponding to this curve is invertible.*

PROOF. We have to prove that the Toeplitz operator with symbol

$$a(\zeta) = \exp\Big(-\imath\frac{2}{p}\arg\mathfrak{c}'(\zeta)\Big)$$

is invertible. Recall that $\arg\mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg\zeta - \dfrac{\pi}{2}$ for $\zeta \in \mathbb{T}$.

The idea of the proof is to represent the symbol in the form $a(\zeta) = c(\zeta)a_0(\zeta)$, where a_0 is p-sectorial and $c \in C(\mathbb{T})$ is such that $\operatorname{ind}_{c(\mathbb{T})}(0) = 0$. To this end, we first choose a continuous branch of the function $\arg\mathfrak{c}'(\zeta)$ on $\mathbb{T}\setminus\{\zeta_1,\dots,\zeta_n\}$, where $z_k = \mathfrak{c}(\zeta_k)$ for $k = 1,\dots,n$. Consider an arc (ζ_1,ζ_1^+) on $\mathbb{T}$ and take the branch of $\arg\mathfrak{c}'(\zeta)$ such that (5.1) holds for $k = 1$. Hence it follows that the argument of $a(\zeta)$ satisfies

$$\sup_{\zeta\in(\zeta_1,\zeta_1^+)}\Big|-\frac{2}{p}\arg\mathfrak{c}'(\zeta) - \psi_1\Big| < \begin{cases} \dfrac{\pi}{p}, & \text{if } p \ge 2,\\[2ex] \dfrac{\pi}{q}, & \text{if } 1 < p < 2,\end{cases} \tag{5.2}$$

where

$$\psi_k = -\frac{2}{p}\varphi_k + \frac{2}{p}\arg\zeta_k + \frac{\pi}{p}$$

for $k = 1,\dots,n$.

Then we extend $\arg\mathfrak{c}'(\zeta)$ to a continuous function on the arc (ζ_1,ζ_2). Note that the right-hand side of (5.2) can be written as

$$\frac{\pi}{\max\{p,q\}}$$

for all $1 < p < \infty$. It is easy to see that there is an integer number j_2 with the property that

$$\sup_{\zeta\in(\zeta_2^-,\zeta_2)}\Big|-\frac{2}{p}\arg\mathfrak{c}'(\zeta) - (\psi_2 + 2\pi j_2)\Big| < \frac{\pi}{\max\{p,q\}}, \tag{5.3}$$

where ψ_2 is defined above. Choose the continuous branch of $\arg\mathfrak{c}'(\zeta)$ on (ζ_2,ζ_2^+), such that (5.3) is still valid with (ζ_2^-,ζ_2) replaced by (ζ_2,ζ_2^+).

We now extend $\arg\mathfrak{c}'(\zeta)$ to a continuous function on the arc (ζ_2,ζ_3), and so on. Proceeding in this fashion, we get a continuous branch of $\arg\mathfrak{c}'(\zeta)$ on all of (ζ_n,ζ_1) satisfying

$$\sup_{\zeta\in(\zeta_1^-,\zeta_1)}\Big|-\frac{2}{p}\arg\mathfrak{c}'(\zeta) - (\psi_1 + 2\pi j_1)\Big| < \frac{\pi}{\max\{p,q\}} \tag{5.4}$$

with some integer j_1.

The task is now to show that $j_1 = 0$, and so the inequality (5.2) actually holds with (ζ_1,ζ_1^+) replaced by $(\zeta_1^-,\zeta_1^+)\setminus\{\zeta_1\}$. For this purpose, we link any two points z_k^- and z_k^+ together by a smooth curve $\mathcal{A}_k$, such that

1) $\tilde{\mathcal{C}} = \big((z_1^+,z_2^-)\cup\ldots\cup(z_n^+,z_1^-)\big)\cup(\mathcal{A}_1\cup\ldots\cup\mathcal{A}_n)$ is a smooth closed curve which bounds a simply connected domain $\tilde{\mathcal{D}}$.

2) The angle $\tilde{\alpha}(z)$ at which the tangent of $\tilde{\mathcal{C}}$ at the point z intersects the real axis satisfies (5.1).

Consider a conformal map $z = \tilde{\mathfrak{c}}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}$. By the very construction, $\tilde{\alpha}(z) = \alpha(z)$ holds for all $z \in (z_1^+, z_2^-) \cup \ldots \cup (z_n^+, z_1^-)$. Suppose (5.4) is valid with $j_1 \neq 0$. Then, in particular,

$$\Big| -\frac{2}{p} \arg \tilde{\mathfrak{c}}'(\tilde{\zeta}_1^-) - (\psi_1 + 2\pi j_1) \Big| < \frac{\pi}{\max\{p,q\}}$$

where $z_1^- = \tilde{\mathfrak{c}}(\tilde{\zeta}_1^-)$. From this we deduce that the function $\arg \tilde{\mathfrak{c}}'(\zeta)$ has a nonzero increment when the point ζ makes one turn along the unit circle $\mathbb{T}$ starting from the point $\tilde{\zeta}_1^+$ with $z_1^+ = \tilde{\mathfrak{c}}(\tilde{\zeta}_1^+)$. Hence it follows, by the argument principle, that the function $\tilde{\mathfrak{c}}'$ has zeros in $\mathbb{D}$, which contradicts the conformality of $\tilde{\mathfrak{c}}$. Thus, $j_1 = 0$ in (5.4).

We have thus chosen a continuous branch of the function $\arg \mathfrak{c}'(\zeta)$ on the set $\mathbb{T} \setminus \{\zeta_1, \ldots, \zeta_n\}$, satisfying

$$\sup_{\zeta \in (\zeta_k^-, \zeta_k^+) \setminus \{\zeta_k\}} \Big| -\frac{2}{p} \arg \mathfrak{c}'(\zeta) - (\psi_k + 2\pi j_k) \Big| < \frac{\pi}{\max\{p,q\}} \tag{5.5}$$

for all $k = 1, \ldots, n$, where j_k is integer and $j_1 = 0$. This allows one to construct the desired factorisation of $a(\zeta)$.

We first define $c(\zeta)$ away from the arcs (ζ_k^-, ζ_k^+) which encompass singular points ζ_k of $\mathfrak{c}'(\zeta)$. Namely, we set

$$c(\zeta) := \exp\Big(-\imath \frac{2}{p} \arg \mathfrak{c}'(\zeta) \Big)$$

for $\zeta \in \mathbb{T} \setminus \bigcup_{k=1}^n (\zeta_k^-, \zeta_k^+)$.

To define $c(\zeta)$ in any arc (ζ_k^-, ζ_k^+) with $k = 1, \ldots, n$, we pick an $\varepsilon_k > 0$ small enough, so that $\arg \zeta_k^- + \varepsilon_k < \arg \zeta_k < \arg \zeta_k^+ - \varepsilon_k$. Then the symbol $c(\zeta)$ is defined by

$$c(\zeta) := \exp\Big(-\imath \frac{2}{p} \frac{(\arg \zeta_k^- + \varepsilon_k - \arg \zeta) \arg \mathfrak{c}'(\zeta_k^-) + (\arg \zeta - \arg \zeta_k^-) \tilde{\varphi}_k}{\varepsilon_k} \Big),$$

if $\zeta \in (\zeta_k^-, e^{\imath \varepsilon_k} \zeta_k^-]$,

$$c(\zeta) := \exp\Big(-\imath \frac{2}{p} \tilde{\varphi}_k \Big),$$

if $\zeta \in (e^{\imath \varepsilon_k} \zeta_k^-, e^{-\imath \varepsilon_k} \zeta_k^+)$, and

$$c(\zeta) := \exp\Big(-\imath \frac{2}{p} \frac{(\arg \zeta_k^+ - \arg \zeta) \tilde{\varphi}_k + (\arg \zeta - \arg \zeta_k^+ + \varepsilon_k) \arg \mathfrak{c}'(\zeta_k^+)}{\varepsilon_k} \Big),$$

if $\zeta \in [e^{-\imath \varepsilon_k} \zeta_k^+, \zeta_k^+)$. Here, $\tilde{\varphi}_k = -\frac{p}{2}(\psi_k + 2\pi j_k)$.

Obviously, $c(\zeta)$ is a nonvanishing continuous function of $\zeta \in \mathbb{T}$. From (5.5) it follows that $\operatorname{ind}_{c(\mathbb{T})}(0) = 0$. Put

$$a_0(\zeta) := \frac{a(\zeta)}{c(\zeta)}$$

for $\zeta \in \mathbb{T}$. Then

$$\arg a_0(\zeta) = 0$$

for all $\zeta \in \mathbb{T} \setminus \bigcup_{k=1}^n (\zeta_k^-, \zeta_k^+)$. Moreover, if the numbers $\varepsilon_1, \ldots, \varepsilon_n$ are small enough then

$$|\arg a_0(\zeta)| \leq \frac{\pi}{\max\{p,q\}}$$

for all $\zeta \in \bigcup_{k=1}^{n}(\zeta_k^-, \zeta_k^+)$. Hence, $a_0(\zeta)$ is a p-sectorial symbol, which yields the desired factorisation.

By Theorem 4.6, 1) we conclude that the Toeplitz operator $T(a_0)$ is invertible in the space H^{p+}. Moreover, Theorem 4.7, shows that $T(a)$ is Fredholm of index zero. Finally, Theorem 4.2 implies that the operator $T(a)$ is actually invertible, as desired. □

Corollary 3.2 gives the solution of the Dirichlet problem in $\mathcal{D}$ via the inverse operator $(T(a)^{-1}$. If $a(\zeta)$ admits a p-factorisation then Corollary 4.3 yields an explicit formula for $(T(a)^{-1}$. In case the boundary of $\mathcal{D}$ is a sectorial curve, it is possible to construct a p-factorisation of $a(\zeta)$ with the help of conformal map $z = \mathfrak{c}(\zeta)$.

THEOREM 5.3. *Let $\mathcal{C}$ be a p-sectorial curve. For any $u_0 \in L^p(\mathcal{C})$, the Dirichlet problem has a unique solution $u = \Re f$ with $f \in E^p(\mathcal{D})$ given by*

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta + \mathfrak{c}^{-1}(z)}{\zeta - \mathfrak{c}^{-1}(z)} u_0(\mathfrak{c}(\zeta)) \frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}$.

PROOF. According to Theorems 5.2 and 4.1, a p-factorisation of the symbol of Toeplitz operator corresponding to the Dirichlet problem in a domain with p-sectorial boundary, if there is any, looks like $a(\zeta) = a^+(\zeta)a^-(\zeta)$. We begin with the representation

$$a(\zeta) = \left(\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}\right)^{1/p}$$

for $\zeta \in \mathbb{T}$, cf. (3.6). In the case of p-sectorial curves the angle $\alpha(z)$ is bounded, and so the curve $\mathcal{C}$ is rectifiable. By a well-known result (see for instance [**Gol69**]), the derivative $\mathfrak{c}'(\zeta)$ belongs to H^{1+}, whence $\sqrt[p]{\mathfrak{c}'(\zeta)} \in H^{p+}$ and $\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \in H^{p-} \oplus \{c\}$. Comparing this with $a(\zeta) = a^+(\zeta)a^-(\zeta)$, we get

$$\sqrt[p]{\mathfrak{c}'(\zeta)}a^+(\zeta) = \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}(1/a^-(\zeta)).$$

By (4.2), the left-hand side of this equality belongs to H^{1+} and the right-hand side to $H^{1-} \oplus \{c\}$. Hence it follows that

$$\begin{array}{rcl} \sqrt[p]{\mathfrak{c}'(\zeta)}a^+(\zeta) & = & c, \\ \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}(1/a^-(\zeta)) & = & c \end{array}$$

where c is a complex constant. The factorisation $a(\zeta) = a^+(\zeta)a^-(\zeta)$ with

$$\begin{array}{rcl} a^+(\zeta) & = & c\left(1/\sqrt[p]{\mathfrak{c}'(\zeta)}\right), \\ a^-(\zeta)) & = & \dfrac{1}{c}\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \end{array}$$

satisfies (4.2), and $(T(a))^{-1} = \sqrt[p]{\mathfrak{c}'(\zeta)}P_{\mathbb{T}}^+(1/\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}})$, which is due to Corollary 4.3. This establishes the theorem when combined with the formula of Corollary 3.2. We fill in details.

We first observe that condition (3.9) is fulfilled. Indeed, from

$$\begin{array}{rcl} \left(T(a)^{-1}1\right)(\zeta) & = & \sqrt[p]{\mathfrak{c}'(\zeta)}P_{\mathbb{T}}^+(1/\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}})(\zeta) \\ & = & \sqrt[p]{\mathfrak{c}'(\zeta)}\left(1/\overline{\sqrt[p]{\mathfrak{c}'(0)}}\right), \end{array}$$

it follows that $\left(T(a)^{-1}1\right)(0) = 1$, for the derivative $\mathfrak{c}'(0)$ is positive. On the other hand, the equality

$$\begin{aligned} \left(T(a)^{-1}f^+\right)(\zeta) &= \sqrt[p]{\mathfrak{c}'(\zeta)}P^+_{\mathbb{T}}\Big((1/\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}})P^+_{\mathbb{T}}\Big(2u_0(\mathfrak{c}(\zeta))\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}\Big)\Big)(\zeta)\\ &= \sqrt[p]{\mathfrak{c}'(\zeta)}P^+_{\mathbb{T}}\Big(2u_0(\mathfrak{c}(\zeta))\Big)(\zeta)\end{aligned}$$

implies that $\Im\left(T(a)^{-1}f^+\right)(0) = 0$, for $\mathfrak{c}'(0) > 0$ and $\Im P^+_{\mathbb{T}}\left(2u_0(\mathfrak{c}(\zeta))\right)(0) = 0$, the latter being due to the fact that u_0 is real-valued. Thus we may use the formula of Corollary 3.2.

An easy computation shows that (see the proof of Corollary 3.2)

$$\begin{aligned} h^+(\zeta) &= \left(T(a)^{-1}f^+\right)(\zeta) - \frac{1}{2}\left(T(a)^{-1}f^+\right)(0)(T(a)^{-1}1)(\zeta)\\ &= \sqrt[p]{\mathfrak{c}'(\zeta)}P^+_{\mathbb{T}}\Big(2u_0(\mathfrak{c}(\zeta))\Big)(\zeta) - \sqrt[p]{\mathfrak{c}'(0)}P^+_{\mathbb{T}}\Big(u_0(\mathfrak{c}(\zeta))\Big)(0)\,\sqrt[p]{\mathfrak{c}'(\zeta)}(1/\overline{\sqrt[p]{\mathfrak{c}'(0)}})\\ &= \sqrt[p]{\mathfrak{c}'(\zeta)}P^+_{\mathbb{T}}\Big(2u_0(\mathfrak{c}(\zeta))\Big)(\zeta) - P^+_{\mathbb{T}}\Big(u_0(\mathfrak{c}(\zeta))\Big)(0)\,\sqrt[p]{\mathfrak{c}'(\zeta)}\end{aligned}$$

holds for almost all $\zeta \in \mathbb{T}$. Writing the projection $P^+_{\mathbb{T}}$ as the Cauchy integral, we get

$$\begin{aligned} h^+(\zeta) &= \frac{\sqrt[p]{\mathfrak{c}'(\zeta)}}{2\pi\imath}\int_{\mathbb{T}}\left(\frac{u_0(\mathfrak{c}(\zeta'))}{\zeta'-\zeta} - \frac{u_0(\mathfrak{c}(\zeta'))}{2\zeta'}\right)d\zeta'\\ &= \sqrt[p]{\mathfrak{c}'(\zeta)}\int_{\mathbb{T}}\frac{1}{2\pi\imath}\frac{\zeta'+\zeta}{\zeta'-\zeta}\,u_0(\mathfrak{c}(\zeta'))\,\frac{d\zeta'}{\zeta'}\end{aligned}$$

for all $\zeta \in \mathbb{D}$. Since

$$u(z) = \Re\,\frac{h^+(\mathfrak{c}^{-1}(z))}{\sqrt[p]{\mathfrak{c}'(\mathfrak{c}^{-1}(z))}},$$

the proof is complete. $\square$

Note that the formula of Theorem 5.3 coincides with formula (1.10) of [**KKP98**, p. 158].

References

[BS90] Albrecht Böttcher and Bernd Silbermann, *Analysis of Toeplitz operators*, Springer-Verlag, Berlin, 1990. MR1071374 (92e:47001)

[DS00] R. Duduchava and B. Silbermann, *Boundary value problems in domains with peaks*, Mem. Differential Equations Math. Phys. **21** (2000), 1–122 (English, with English and Georgian summaries). MR1818729 (2001m:35082)

[Esk73] G. I. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Operators*, Nauka, Moscow, 1973.

[GK92] I. Gokhberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations*, vols. 1,2, Birkhäuser, Basel, 1992.

[Gol69] G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, R.I., 1969. MR0247039 (40 #308)

[GT12] S. Grudsky and N. Tarkhanov, *Conformal reduction of boundary problems for harmonic functions in a plane Domain with Strong Singularities on the Boundary*, Preprints Inst. Math. Univ. Potsdam **1** (2012), no. 10, 1–51.

[IL98] A. I. Ibragimov and E. M. Landis, *Zaremba's problem for elliptic equations in the neighborhood of a singular point or in the infinity. II*, Appl. Anal. **69** (1998), no. 3-4, 333–347, DOI 10.1080/00036819808840666. MR1706530 (2000g:35036)

[Kel66] M. V. Keldysh, *On the solvability and stability of the Dirichlet problem*, Amer. Math. Soc. Transl. **51** (1966), 1–73.

[Ken94] Carlos E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. 83, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994. MR1282720 (96a:35040)

[KKP98] Givi Khuskivadze, Vakhtang Kokilashvili, and Vakhtang Paatashvili, *Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings*, Mem. Differential Equations Math. Phys. **14** (1998), 195 (English, with English and Georgian summaries). MR1660894 (2000c:30079)

[KP07] G. Khuskivadze and V. Paatashvili, *Zaremba's problem in Smirnov class of harmonic functions in domains with piecewise-Lyapunov boundaries*, Proc. A. Razmadze Math. Inst. **143** (2007), 79–94 (English, with English and Georgian summaries). MR2324507 (2008c:35039)

[KM94] T. Kilpäinen and J. Malý, *The Wiener test for potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.

[KP03] V. Kokilashvili and V. Paatashvili, *The Dirichlet problem for harmonic functions with boundary values from Zygmund classes*, Georgian Math. J. **10** (2003), no. 3, 531–542. Dedicated to the 100th birthday anniversary of Professor Victor Kupradze. MR2023272 (2004k:30093)

[KL91] V. A. Kondratiev and E. M. Landis, *Qualitative theory of second order linear partial differential equations*, In: Encyclopaedia of Mathematical Sciences, vol. 32, Springer-Verlag, 1991, 87–192.

[KMR00] V. A. Kozlov, V. G. Maz′ya, and J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, vol. 85, American Mathematical Society, Providence, RI, 2001. MR1788991 (2001i:35069)

[LS87] Georgii S. Litvinchuk and Ilia M. Spitkovskii, *Factorization of measurable matrix functions*, Operator Theory: Advances and Applications, vol. 25, Birkhäuser Verlag, Basel, 1987. Translated from the Russian by Bernd Luderer; With a foreword by Bernd Silbermann. MR1015716 (90g:47030)

[MNP00] V. Maz'ya, S. Nazarov and B. Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains*, vols. 1, 2, Birkhäuser, Basel, 2000.

[MS89] V. G. Maz′ya and A. A. Solov′ëv, *An integral equation of the Dirichlet problem in a plane domain with cusps on the boundary*, Mat. Sb. **180** (1989), no. 9, 1211–1233, 1296 (Russian); English transl., Math. USSR-Sb. **68** (1991), no. 1, 61–83. MR1017822 (90k:45010)

[Mus68] Muskhelishvili, N. I., *Singulyarnye integralnye uravneniya*, Third, corrected and augmented edition, Izdat. "Nauka", Moscow, 1968 (Russian). Granichnye zadachi teorii funktsii i nekotorye ikh prilozheniya k matematicheskoi fizike. [Boundary value problems in the theory of function and some applications of them to mathematical physics]; With an appendix by B. Bojarski. MR0355495 (50 #7969)

[Rab99] V. Rabinovich, *Potential type operators on curves with vorticity points*, Z. Anal. Anwendungen **18** (1999), no. 4, 1065–1081. MR1736254 (2000m:31002)

[RST04] V. Rabinovich, B.-W. Schulze, and N. Tarkhanov, *Boundary value problems in oscillating cuspidal wedges*, Rocky Mountain J. Math. **34** (2004), no. 4, 1399–1471, DOI 10.1216/rmjm/1181069808. MR2095585 (2005f:35346)

[Vek42] Ilja Vekua, *On a linear boundary value problem of Riemann*, Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11** (1942), 109–139 (Russian, with Georgian summary). MR0011128 (6,123f)

[Zar10] S. Zaremba, *Sur un problème mixte relatif à l'équation de Laplace*, Bulletin de l'Academie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles, série A (1910), 313–344.

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