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Asymptotics of European Double-Barrier Option with Compound Poisson Component

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Abstract

We consider standard European as well as double-barrier European options for underlyings that are given by the superposition of a Guassian and a compound Poisson (jump) process with discrete values. We derive a formula for calculating such options and furthermore show that as the barriers tend to $\pm\infty$, the value of the double-barrier option tends asymptotically to that of the standard option. Numerical examples are provided.

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1 Introduction

The problem of determining the price of a double barrier option when the stock price is modeled by geometric Brownian motion is considered in [15, 18, 19, 25, 32, 37, 38]. In

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[18, 19, 32, 38] the approach is to solve the Black-Scholes partial differential equation on a strip of finite width. However, for many situations geometric Brownian motion is not an adequate model for stock price, and in recent years Lévy processes have come to be used as models for logarithmic stock price. In this context European options [2, 10, 28, 29, 33, 34], perpetual American options [6, 7, 30], and single barrier options [6, 7, 8, 30] have been examined in detail. Recent papers concerning double barrier options under Lévy processes include [4, 5, 9, 14, 35].

In [3] European double-barrier options were considered whose underlyings are Lévy processes formed by the superposition of a Gaussian and a compound Poisson process with discrete values. The determination of the price of such options leads to a Black-Scholes system which is perturbed by a Toeplitz matrix. On the basis of this observation, an effective algorithm was designed for the computation of the price.

This article is a continuation of the investigation in [3]. In the framework of that market model, we derive a calculable formula (Theorem 4.2) for the value of the European option. We further consider the asymptotics of the price of the double barrier option when the upper and lower barriers tend to $\pm\infty$. We construct and justify asymptotic expansions in which, as might be expected, the main term is the price of the standard European option (without barriers). Numerical examples are provided.

This problem was considered in [3] with barriers which were fixed. The algorithm derived there was based on the calculation of eigenvalues and eigenvectors of a system of differential equations. That algorithm is effective when the quotient s^+/s^- of the barriers (or logarithmically, the difference $x^+ - x^-$ in the notation of Section 2 below) is not very large. In contrast, in the present work we consider the case of $x^+ - x^-$ large. We obtain and justify asymptotic formulas when $x^- \to -\infty$ ($s^- \to 0$) and $x^+ \to +\infty$ ($s^+ \to \infty$).

It is well known that the problem of defining the option price $u(x,t)$ in the right way can be delicate. Under our assumptions, we do not have a complete market. As a result, in general an equivalent martingale measure (EMM), which is essential for the valuation of options in this context, is not unique [6, p. 97]. A convenient EMM is produced by the Esscher transform [6, pp. 98–99] with very little calculation. However, our formulas for option evaluation can be used with any preferred EMM; naturally the results produced will depend to some degree on this choice.

In Section 2 we give the necessary background details on the market model and describe the EMM. In Section 3 we describe the Black-Scholes system corresponding to the knock-out double barrier option problem and give the existence result for this system.[1] Section 4 describes the problem of the European option (i.e. without barriers) and includes a calculable formula for its exact solution. Section 5 is devoted to obtaining and justifying the asymptotic expansion of the double barrier option price in the framework of the model. Section 6 specializes the results for double barriers to the classical case for which the market has no jumps. Finally, numerical aspects of the algorithm are discussed in Section 7.

[1] The classical Black-Scholes fórmula, which refers specifically to put and call options, is technically excluded from our discussion since it involves a payoff which is not in $L_2(\mathbb{R})$.

2 Market Model

2.1 Notation and terminology

The mathematical setting will be that of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}, P)$ on which $\{X_\tau\}_{\tau\geq 0}$ is the Lévy process [36, p. 202] specified by its characteristic exponent $\psi(\xi)$,

$$\begin{aligned} \mathbf{E}^P(e^{i\xi X_\tau}) &= e^{-\tau\psi(\xi)}, \\ \psi(\xi) &= \frac{\tilde{\sigma}^2}{2}\xi^2 - i\tilde{\mu}\xi + \tilde{\lambda}\Big(1 - \sum_{j=-\infty}^{\infty} \tilde{p}_j e^{i\tilde{y}_j\xi}\Big). \end{aligned} \tag{2.1}$$

Here $\mathbf{E}^P$ denotes the expected value taken with respect to the probability measure P. The parameters $\tilde{\sigma}, \tilde{\mu}, \tilde{\lambda}, \tilde{p}_j, \tilde{y}_j$ are real numbers subject to the constraints $\tilde{\sigma} > 0, \tilde{\mu} \in \mathbb{R}, \tilde{\lambda} \geq 0, \tilde{p}_j \geq 0$, $\sum_j \tilde{p}_j = 1$. We consider $\{X_\tau\}_{\tau\geq 0}$ under the assumption that we are given two absorbing barriers, located at $x^\pm$ where $x^- < x^+$. Let $g(x)$ be the payoff function, satisfying $g \in L_2(\mathbb{R})$.

We denote by T_0 the purchase date of the option, while the expiration date is $T_1 = T_0 + t$. Our objective is to compute the expected value of the discounted payoff $e^{-rt}g(X_{T_1})$ with respect to an equivalent martingale measure (EMM) Q for P under the condition that X_{T_0} is known to be a given value x in the interval (x^-, x^+). Thus, we look for the quantity

$$U(x, T_0, T_1) = \mathbf{E}^Q\Big[e^{-r(T_1-T_0)}g(X_{T_1})\mathbf{1}_{\eta>T_1}|_{\mathcal{F}_{T_0}} X_{T_0} = x\Big],$$

where $\mathbf{1}_{(\cdot)}$ denotes the characteristic function of a set and where the hitting time η is the random variable

$$\eta = \inf\{\tau > 0 :\ X_\tau \in (-\infty, x^-] \text{ or } X_\tau \in [x^+, \infty)\}.$$

We think of T_0 as being fixed. Then the quantity $U(x, T_0, T_1)$ is a function of only x and of $t = T_1 - T_0$, and therefore we henceforth consider the function

$$u(x,t) = \mathbf{E}^Q\Big[e^{-rt}g(X_{T_0+t})\mathbf{1}_{\eta>T_0+t}\Big|_{\mathcal{F}_{T_0}} X_{T_0} = x\Big]. \tag{2.2}$$

Thus the value $S_0u(x,t)$ may be interpreted as the price for a knock-out double-barrier option. We interpret

$$S_\tau = S_0 e^{X_\tau} \tag{2.3}$$

as the market price of a stock at time τ. The market drift and volatility are $\tilde{\mu}$ and $\tilde{\sigma}$, while the parameter r is the rate of interest of the riskless asset (bond). Fix $t > 0$ and let $X_0 = x$. At time $\tau = 0$ the holder pays the premium $u(x,t)$ to the writer and at time $\tau = t$ receives in return the payoff $h(S_t) := S_0 e^{g(X_t)}$ from the writer provided that the barrier condition $x^- < X_\tau < x^+$, i.e., $s^- < S_\tau/S_0 < s^+$ (where $s^\pm = e^{x^\pm}$), is maintained for all $\tau \in [0,t]$.

Our assumptions on the market say that

$$X_\tau = \tilde{\sigma}^2 W_\tau + \tilde{\mu}\tau + \sum_{k=1}^{N_\tau} Y_k \tag{2.4}$$

where $W_\tau \sim N(0, \sqrt{\tau})$ is normalized Brownian motion, $\tilde{\mu}$ characterizes the drift, N_τ is the Poisson process at rate $\tilde{\lambda}$,

$$P(N_\tau = \tilde{y}_k) = \frac{(\tilde{\lambda}\tau)^k}{k!}e^{-\tilde{\lambda}\tau}, \quad k = 0, 1, 2, \ldots, \tag{2.5}$$

and $Y_1, Y_2, \ldots$ are independent identically distributed random variables with

$$P(Y_k = \tilde{y}_j) = \tilde{p}_j, \quad j = 0, \pm 1, \pm 2, \ldots \tag{2.6}$$

As was mentioned in the introduction, the choice of an EMM is a delicate matter which can influence the option valuation in an incomplete market. In [12] a specific formula, known as the Esscher transform, was introduced which is easy to calculate and which has been used since in many investigations on option pricing, appearing in standard texts such as [6]. In the present work it does not matter whether or not the Esscher transform is the particular EMM chosen. We will simply assume that Q is an EMM for P determined by some parameters

$$\sigma, \mu, \lambda, p_j, y_j$$

satisfying $\sigma > 0$, $\mu \in \mathbb{R}$, $\lambda \geq 0$, $p_j \geq 0$, $\sum p_j = 1$, and $y_j = \tilde{y}_j$. The characteristic function ψ^Q of Q is given analogously to (2.1) by

$$\psi^Q(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\Big(1 - \sum_{j=-\infty}^{\infty} p_j e^{iy_j\xi}\Big). \tag{2.7}$$

2.2 Almost-periodic part of Lévy process

Now we calculate the density function $\rho_Q(y)$ corresponding to the martingale measure Q. According to the Lévy-Khintchine formula for the EMM Q [6, p. 105] we have that

$$\rho_Q(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t\psi^Q(\xi)+i\xi y}\, d\xi.$$

We denote by $\Pi_W(\mathbb{R})$ the class of functions of the form

$$J(\xi) = \sum_{j=-\infty}^{\infty} c_j e^{i\sigma_j\xi}, \quad \xi \in \mathbb{R}, \tag{2.8}$$

where $\sigma_j \in \mathbb{R}$, $c_j \in \mathbb{C}$, and $\sum_{j=-\infty}^{\infty} |c_j| < \infty$. This class $\Pi_W(\mathbb{R})$ is called the *class of Wiener almost-periodic functions* and is an algebra of functions. Moreover, if the function $F(z)$ is analytic on a neighborhood of the closure of the range of the function $J(\xi)$, then $F(J(\xi)) \in \Pi_W(\mathbb{R})$ (see [26]).

The almost-periodic part of the characteristic exponent $\psi^Q(\xi)$ of Q defined in (2.7) is [17]

$$J_0(\xi) := \lambda\Big(1 - \sum_{j=-\infty}^{\infty} p_j e^{iy_j\xi}\Big), \quad y_j \in \mathbb{R}, \tag{2.9}$$

where $p_j \geq 0$, $\sum_{j=-\infty}^{\infty} p_j = 1$ and $J_0 \in \Pi_W(\mathbb{R})$. By the above remarks, we have a representation

$$e^{-tJ_0(\xi)} = \sum_{j=-\infty}^{\infty} c_j(t) e^{i\sigma_j\xi}, \quad \sigma_j \in \mathbb{R}, \tag{2.10}$$

with coefficients in $l_1(\mathbb{C})$, that is

$$\sum_{j=-\infty}^{\infty} |c_j(t)| < \infty. \tag{2.11}$$

Therefore we can write

$$\begin{aligned}
\rho_Q(y) &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} c_j(t)\int_{-\infty}^{\infty} e^{-t(\sigma^2/2)\xi^2+i(\mu t+y+\sigma_j)\xi}\,d\xi \\
&= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} c_j(t)e^{-(\mu t+y+\sigma_j)^2/(2\sigma^2 t)}\times \\
&\qquad \int_{-\infty}^{\infty}\exp\left\{\frac{-\sigma^2 t}{2}\left[\xi - i\left(\frac{\mu t+y+\sigma_j}{\sigma^2 t}\right)\right]^2\right\}d\xi \\
&= \frac{\sqrt{2}}{2\pi\sigma\sqrt{t}}\sum_{j=-\infty}^{\infty} c_j(t)e^{-(\mu t+y+\sigma_j)^2/(2\sigma^2 t)}\int_{-\infty}^{\infty} e^{-\xi'^2}\,d\xi',
\end{aligned}$$

so

$$\rho_Q(y) = \frac{1}{\sigma\sqrt{2\pi t}}\sum_{j=-\infty}^{\infty} c_j(t)e^{-(\mu t+y+\sigma_j)^2/(2\sigma^2 t)}. \tag{2.12}$$

From this we can see that the measure Q has the form

$$Q(A) = \int_A \rho_Q(y)\,dy,$$

where A is any Borel set of $\mathbb{R}$, and ρ_Q is given by (2.12) and thus is absolutely continuous with respect to Lebesgue measure.

3 The Generalized Black-Scholes Equation

We derive here the partial differental equation (heat equation) which describes solutions to the option problem as a function of x and t.

Let $\sigma > 0$ and $r > 0$, and let μ, λ, p_j be the EMM market parameters with y_j as in (2.5), (2.6). Let $x^- < x^+$ and write $I^* = (x^-, x^+)$; we will also abbreviate $x^* = \min(|x^-|, |x^+|)$ when convenient.

3.1 Heat equation for Lévy market

Consider the operator A defined by

$$\begin{aligned}
(Af)(x) &:= -\frac{\sigma^2}{2}f''(x) - \mu f'(x) + rf(x) + \lambda f(x) \\
&\quad - \lambda\sum_{j=-\infty}^{\infty} p_j f(x+y_j)\mathbf{1}_{I^*}(x+y_j).
\end{aligned} \tag{3.1}$$

We think of A as an operator on $L_2(I^*)$ with the (dense) domain $\mathcal{D}(A) = C^2(\operatorname{clos} I^*)$. In [6] it is shown that the function given by (2.2) satisfies the generalized Black-Scholes equation

$$u_t(x,t) + (Au)(x,t) = 0, \quad (x,t) \in I^* \times (0,\infty), \tag{3.2}$$

where A is taken in the variable x, along with the boundary conditions

$$u(x,0) = g(x), \quad x \in I^*, \tag{3.3}$$

while for $t > 0$ the function $u(x,t)$ is continuous in $x \in \mathbb{R}$ and

$$u(x,t) = 0, \quad (x,t) \in (I^*)^c \times (0,\infty). \tag{3.4}$$

Here $(I^*)^c := \mathbb{R} \setminus I^*$ is the region outside the barrier.

More exactly, Theorem 2.13 of [6, p. 65] applies when the Lévy process satisfies the so-called ACP condition [6, p. 59]. For us, when $\sigma \neq 0$, the remarks at the end of the previous section show that the measure Q is absolutely continuous with respect to Lebesgue measure. Therefore by Lemma 2.4 of [6], the ACP condition holds.

Condition (3.4) is in fact superfluous because we consider A as acting only on $L_2(I^*)$. We may also write (3.2)–(3.4) in the form

$$\begin{aligned} u_t(x,t) &= \frac{\sigma^2}{2} u_{xx}(x,t) + \mu u_x(x,t) - (r+\lambda)u(x,t) \\ &\quad + \lambda \sum_{j=-\infty}^{\infty} p_j u(x+y_j,t)\mathbf{1}_{I^*}(x+y_j) \end{aligned} \tag{3.5}$$

on $I^* \times (0,\infty)$ with boundary condition

$$u(x,0) = g(x) \text{ for } x \in I^*, \tag{3.6}$$

while we have $u(\cdot,t) \in C^0(\operatorname{clos} I^*)$ satisfying

$$u(x^-,t) = u(x^+,t) = 0 \text{ for } t \in (0,\infty). \tag{3.7}$$

3.2 Associated Cauchy problem

For $t \in [0,\infty)$, we define $\widetilde{u}(t) \in L_2(I^*)$ by $(\widetilde{u}(t))(x) := u(x,t)$. Then problem (3.2)–(3.3) can be interpreted as the Cauchy problem

$$\frac{d}{dt}\widetilde{u}(t) = -A(\widetilde{u}(t)), \quad \widetilde{u}(0) = g \tag{3.8}$$

in which the operator A is understood as having (dense) domain $C_0^2(\operatorname{clos} I^*)$, the subspace of $C^2(\operatorname{clos} I^*)$ of functions vanishing at $x^\pm$.

Theorem 3.1. *Let A be the operator* (3.1). *Problem* (3.8) *is well-posed in the sense that $-A$ generates a C^0-contraction semigroup and*

$$\|e^{-tA}g\|_2 \le e^{-rt}\|g\|_2.$$

The resolvent operator $(\lambda 1 + A)^{-1}$ is compact and hence the spectrum of $-A$ consists entirely of isolated eigenvalues of finite algebraic multiplicity.

Proof. Writing $D_x = d/dx$, we have

$$-A = \frac{\sigma^2}{2} D_x^2 + \mu D_x - r1 - \lambda(1 - V) \tag{3.9}$$

where $(Vf)(x) := \sum_j p_j f(x+y_j) \mathbf{1}_{I^*}(x+y_j)$. Clearly, the desired inequality will follow once we have shown that $(\sigma^2/2)D_x^2 + \mu D_x - \lambda(1-V)$ generates a C^0-contraction semigroup. By [3, Theorem 2] and [16, Theorem 2.6.1], it suffices to show that $-\lambda(1-V)$ is bounded and dissipative. The boundedness is obvious. To show that $-\lambda(1-V)$ is dissipative, let $\mathcal{F}$ denote the Fourier transform,

$$(\mathcal{F} f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

($\xi \in \mathbb{R}$), and note that $-\lambda(1-V)$ can be written as $-\lambda \mathcal{F} J_0 \mathcal{F}^{-1}$ with $J_0(\xi)$ defined by (2.9). Since

$$\begin{aligned} \operatorname{Re}\langle -\lambda(1-V)f, f\rangle &= -\lambda \operatorname{Re}\langle \mathbf{1}_{I^*} \mathcal{F} J_0 \mathcal{F}^{-1} f, f\rangle \\ &= -\lambda \operatorname{Re}\langle \mathcal{F} J_0 \mathcal{F}^{-1} f, f\rangle = -\lambda \operatorname{Re}\langle J_0 \mathcal{F}^{-1} f, \mathcal{F}^{-1} f\rangle \\ &\le 0 \end{aligned}$$

(recall that $\operatorname{Re} J_0 \ge 0$), we see that $-\lambda(1-V)$ is dissipative.

Finally, since $(\sigma^2/2)D_x^2 + \mu D_x - r1$ has compact resolvent and using [22, p. 187], we see by (3.9) that $-A$ differs from $(\sigma^2/2)D_x^2 + \mu D_x - r1$ by a bounded operator, and deduce that $-A$ must also have a compact resolvent [22, p. 214]. □

Consider the non-homogeneous Cauchy problem corresponding to problem (3.8),

$$\frac{d}{dt}\tilde{u}(t) = -(A\tilde{u})(t) + \tilde{f}(t), \quad \tilde{u}(0) = g, \tag{3.10}$$

where for each t, the function $\tilde{f}(t)(x) = f(t,x)$ is in $L_2(I^*)$, and $\tilde{f}(t)$ varies continuously in t as an element of $L_2(I^*)$.

Let $\tilde{u}_0(t) = e^{-tA} g$ be the solution of problem (3.8). Then by [24, Section 61] the solution of (3.10) is unique and has the form

$$\tilde{u}(t) = \tilde{u}_0(t) + \int_0^t \tilde{u}_0(t-\tau)\tilde{f}(\tau)\,d\tau.$$

This produces the function $u(x,t) = \tilde{u}(t)(x)$, which according to Theorem 3.1 satisfies the following L_2-estimate, which we will use in Section 5:

$$\begin{aligned} \|u(\cdot,t)\|_2 &\le e^{-rt}\|g\|_2 + \int_0^t e^{-r(t-\tau)} \|f(\cdot,\tau)\|_2\,d\tau \\ &\le e^{-rt}\|g\|_2 + \int_0^t \|f(\cdot,\tau)\|_2\,d\tau. \end{aligned} \tag{3.11}$$

4 European Option with a Compound Poisson Component

In this section we derive a calculable formula (Theorem 4.2) for the value of a European option on an underlying driven by a Lévy process with jump discontinuities. Furthermore, we prove growth estimates on the value of this option as a funcion of barriers tending simultaneously to $\pm\infty$. These estimates will be needed to justify the asymptotic formulas of sections 5 and 6.

4.1 Formula for European option

Consider the standard European style option (i.e., no barriers) in the particular case of the time-τ stock price of the form (2.3) where X_τ is a Lévy process satisfying (2.4)–(2.6) with characteristic exponent of the form (2.1). Analogously to (2.2) we look now for the quantity

$$u_\infty(x,t) = \mathbf{E}^Q[\, e^{-rt} g(X_{T_0+t})|_{\mathcal{F}_{T_0}} X_{T_0} = x \,] \tag{4.1}$$

where Q is an EMM for P. The value $S_0 u_\infty(x,t)$ may be interpreted as the price of a European option with payoff $g(x)$ at expiry date $T_1 = T_0 + t$. At time $\tau = 0$ the holder pays a premium of $u(x,t)$ to the writer and at time $\tau = t$ receives in return the amount

$$h(S_t) = S_0 e^{g(X_t)}.$$

According to [6, Ch. 4, p. 105] the value (4.1) can be represented in the form

$$u_\infty(x,t) = \frac{e^{-rt}}{2\pi} \int_{-\infty}^{\infty} e^{-t\psi^Q(\xi)+i\xi x}\, \hat{g}(\xi)\, d\xi \tag{4.2}$$

where $\hat{g} = \mathcal{F} g$ is again the Fourier transform of g. We thus have that

$$u_\infty(x,t) = e^{-rt} \left(\mathcal{F}^{-1} e^{-t\psi^Q(\cdot)} \mathcal{F} g\right)(x,t). \tag{4.3}$$

Theorem 4.1. *Let $g \in L_2(\mathbb{R})$. Then for $t > 0$,*

$$\|u_\infty(\cdot,t)\|_{L_2(\mathbb{R})} \le e^{-rt} \|g\|_{L_2}$$

and

$$\lim_{x^\pm \to \pm\infty} \|u_\infty(\cdot,\tau)\|_{L_2((I^*)^c)} = 0$$

uniformly in the interval $\tau \in [\tau_0, t]$ for any τ_0 between 0 and t.

Proof. From the Lévy-Khintchine for Q we have that $\mathrm{Re}\,\psi^Q(\xi) \ge 0$ for all $\xi \in \mathbb{R}$, consequently $\|e^{-t\psi^Q(\cdot)} \mathcal{F} g\|_{L_2} \le \|\mathcal{F} g\|_{L_2}$. It is well known that $\|\mathcal{F} g\|_{L_2} = \|g\|_{L_2}$ for an arbitrary function $g \in L_2(\mathbb{R})$, so the norm inequality on $L_2(\mathbb{R})$ is verified. Consider now the function

$$a(\tau, x^-, x^+) := e^{-\tau t} \|u_\infty(\cdot,\tau)\|_{L_2((I^*)^c)}$$

(where t is). This function is continuous in τ on the interval $[\tau_0, t]$ by (4.3) and for any fixed $\tau \in [\tau_0, t]$,

$$\lim_{x^\pm \to \pm\infty} a(\tau, x^-, x^+) = 0.$$

Since g is in $L_2(\mathbb{R})$, we have uniform convergence as claimed. □

Recalling (2.9),(2.10), we can now rewrite (4.2) as

$$u_\infty(x,t) = \frac{e^{-rt}}{2\pi}\sum_{j=-\infty}^{\infty} c_j(t)\int_{-\infty}^{\infty} e^{ix\xi}\Big(e^{-\frac{\sigma^2}{2}t\xi^2}\Big)\Big(e^{i(\mu t+\sigma_j)\xi}\hat{g}(\xi)\Big)\,d\xi.$$

By properties of the Fourier transform of the product, this becomes

$$u_\infty(x,t) = e^{-rt}\sum_j c_j(t)\int_{-\infty}^{\infty} g(\xi+\mu t+\sigma_j)\gamma(\xi-x)\,d\xi$$

where

$$\begin{aligned}\gamma(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi v}e^{-\frac{\sigma^2 t}{2}v^2}\,dv\\ &= \frac{e^{-\frac{\xi^2}{2\sigma^2 t}}}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 t}{2}(v-\frac{i\xi}{\sigma^2 t})^2}\,dv\\ &= \frac{\sqrt{2}}{2\pi\sigma\sqrt{t}}e^{-\frac{\xi^2}{2\sigma^2 t}}\int_{-\infty}^{\infty} e^{-v'^2}\,dv',\end{aligned}$$

so

$$\gamma(\xi) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-\xi^2/(2\sigma^2 t)}.$$

Recall that t is strictly positive in the above reasoning. We have proved the following theorem, which shows how we can recover $u_\infty(x,t)$ if we know the numbers σ_j and $c_j(t)$. But it is easy to see that $\{\sigma_j\}$ is the subgroup of $\mathbb{R}$ generated by $\{y_j\}$, while $c_j(t)$ is given by the formula

$$c_j(t) = \lim_{N\to\infty}\frac{1}{2N}\int_{-N}^{N} e^{-tJ_0(\xi)-i\sigma_j\xi}\,d\xi. \tag{4.4}$$

Theorem 4.2. *The fair price $u_\infty(x,t)$ of a European option with payoff $g(x)$ under the Lévy process X_τ is given by the formula*

$$u_\infty(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-rt}\sum_{j=-\infty}^{\infty} c_j(t)\int_{-\infty}^{\infty} g(\xi+\mu t+\sigma_j)e^{-(\xi-x)^2/(2\sigma^2 t)}\,d\xi \tag{4.5}$$

for $x\in\mathbb{R}$ and $t>0$, where the coefficients c_j and σ_j are determined by (2.10) *and* (4.4).

If the function $J_0(\xi)$ is a-periodic, i.e., if

$$J_0(\xi) = \lambda\Big(1-\sum_{j=-\infty}^{\infty} p_j e^{i(2\pi/a)j\xi}\Big), \tag{4.6}$$

then $e^{-tJ_0(\xi)}$ is also a-periodic; that is, formula (4.5) takes the form

$$u_\infty(x,t) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-rt}\sum_{j=-\infty}^{\infty} c_j(t)\int_{-\infty}^{\infty} g(\xi+\mu t+\frac{2\pi}{a}j)e^{-(\xi-x)^2/(2\sigma^2 t)}\,d\xi$$

with

$$c_j(t) = \frac{1}{a}\int_0^a e^{-tJ_0(\xi)-i(2\pi/a)j\xi}\,d\xi. \tag{4.7}$$

The rate of convergence of the series (2.11) under the condition (4.6) depends on properties of the function $J_0(\xi)$. If $J_0(\xi) \in C^k[0,a]$, that is $J_0^{(k)}(\xi) \in C^0[0,a]$, then

$$|c_j(t)| \le C|j|^{-k}.$$

If the associated function $\widetilde{J}_0(z) := \lambda\left(1 - \sum_j p_j z^j\right)$ is analytic in the annulus

$$\{z \in \mathbb{C}:\ \rho \le |z| \le \frac{1}{\rho}\},$$

for some given ρ, $0 < \rho < 1$, then in fact

$$|c_j(t)| \le C_\rho \rho^{-j}$$

for some $C_\rho > 0$ depending on ρ. When $J_0(\xi)$ is a (trigonometric) polynomial of the form

$$J_0(\xi) = \lambda\Big(1 - \sum_{j=-N}^{N} p_j e^{ij(2\pi/a)\xi}\Big), \tag{4.8}$$

then $\tilde{J}_0(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ and so the growth estimate on $c_j(t)$ holds for all $\rho \in (0,1)$.

For the corresponding value of the European option in a market without jumps, see equation (6.25) below.

4.2 Payoff with compact support

Now let g have compact support. Then it is possible to be more precise regarding the rate of convergence in Theorem 4.1. This is because the Fourier transform $\hat{g}(\xi)$ is now an entire function of exponential type (see [1]) whose derivatives are bounded and square-integrable on horizontal lines:

$$x \mapsto \hat{g}^{(k)}(x + i\rho) \in L_\infty(\mathbb{R}) \cap L_2(\mathbb{R}) \tag{4.9}$$

for every $\rho \in \mathbb{R}$.

Theorem 4.3. *Let g be a function in $L_2(\mathbb{R})$ and of compact support. If the almost-periodic function $J_0(\xi)$ of* (2.9) *has bounded k-th derivative, then for $|x^\pm|$ large enough,*

$$\|u_\infty(\cdot,t)\|_{L_2((I^*)^c)} \le C_k(|x^-|^{-k} + |x^+|^{-k}) \tag{4.10}$$

where $C_k > 0$ does not depend on $x^\pm$ or t.

Let $\rho > 0$. If $J_0(\xi)$ admits analytic continuation to the horizontal strip

$$\{\xi \in \mathbb{C}:\ |\mathrm{Im}\,\xi| \le \rho\}$$

in such a way that

$$\sup_{|\mathrm{Im}\,\xi| \le \rho} \mathrm{Re}\, J_0(\xi) < \infty, \tag{4.11}$$

then

$$\|u_\infty(\cdot,t)\|_{L_2((I^*)^c)} < C_\rho\left(e^{-\rho|x^-|} + e^{-\rho|x^+|}\right), \tag{4.12}$$

where C_ρ does not depend on t or x^-, x^+.

If $J_0(\xi)$ is an almost-periodic polynomial, i.e., if

$$J_0(\xi) = \lambda\Big(1 - \sum_{j=-M}^{M} p_j e^{iy_j\xi}\Big),$$

then the estimate (4.12) holds for all $\rho > 0$.

Proof. We have that $\hat{g}(\xi)$ is an entire function and condition (4.9) holds, so when $J_0(\xi)$ has bounded k-th derivative we can integrate (4.2) by parts k times to obtain

$$u_\infty(x,t) = \frac{e^{-rt}}{2\pi}\frac{1}{(ix)^k}\int_{-\infty}^{\infty}\Big(e^{-t\psi^Q(\xi)}\hat{g}(\xi)\Big)^{(k)} e^{i\xi x}\,d\xi,$$

where $\Big(e^{-t\psi^Q(\xi)}\hat{g}(\xi)\Big)^{(k)} \in L_2(\mathbb{R})$. Analogously to the proof of Theorem 4.1, one verifies statement (4.10).

Now let (4.11) hold. Then it is possible to shift the contour of integration vertically in (4.2), and we have for $x > 0$

$$\begin{aligned} u_\infty(x,t) &= \frac{e^{-rt}}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} e^{-t\psi^Q(\xi)+i\xi x}\hat{g}(\xi)\,d\xi \\ &= \frac{e^{-rt}}{2\pi}e^{-\rho x}\int_{-\infty}^{\infty} e^{-t\psi^Q(\xi'+i\rho)+i\xi' x}\hat{g}(\xi'+i\rho)\,d\xi' \end{aligned}$$

Thus according to (4.9) there is a bound

$$\|u_\infty(\cdot,t)\|_{L_2(x^+,\infty)} \le C_\rho e^{-\rho x^+}$$

for some constant C_ρ.

One treats the case $x < 0$ analogously and thus (4.12) is verified. If $J_0(\xi)$ is an almost-periodic polynomial, then (4.12) holds for every $\rho > 0$ and the last statement of the theorem is proved. □

4.3 x-derivative of European option

Let us suppose $x^+ - x^- > 2$ and decompose the interval I^* of the barrier as the union of the shortened interval

$$I_1^* = (x^- + 1, x^+ - 1)$$

and the ends

$$(I_1^*)^c := I^* \setminus I_1^* = (x^-, x^- + 1] \cup [x^+ - 1, x^+).$$

We will need estimates on the L_2-norms of the x-derivative of $u_\infty(x,t)$ on the set $(I_1^*)^c$, as given in the results below.

Theorem 4.4. *Let $g \in L_2(\mathbb{R})$. Then for $|x^\pm|$ large enough, the European option u_∞ satisfies*

$$\Big\|\frac{\partial}{\partial x}u_\infty(x,\tau)\Big\|_{L_2((I_1^*)^c)} \le C_0\left(\frac{\exp\{-\frac{(x^*/2)^2}{2\sigma^2\tau}\}}{x^*\tau} + \frac{b(x^*)}{\tau^{3/4}}\right),$$

where C_0 is a constant, $x^* := \min(|x^-|, |x^+|)$, and $b(x)$ is a function which does not depend on $\tau \in (0,t]$, satisfying

$$\lim_{x\to\infty} b(x) = 0.$$

Proof. From (4.5) one may calculate that

$$\begin{aligned} \frac{\partial}{\partial x} u_\infty(x,\tau) &= \frac{1}{\sigma^3\sqrt{2\pi\tau^3}} e^{-r\tau} \sum_{j=-\infty}^{\infty} c_j(\tau) \int_{-\infty}^{\infty} g(\xi+\mu t+\sigma_j) \\ &\quad \times (\xi - x) e^{-(\xi-x)^2/(2\sigma^2\tau)}\, d\xi. \end{aligned} \tag{4.13}$$

Introduce the function

$$F(x,\tau) := \int_{-\infty}^{\infty} f(\xi)(\xi-x)e^{-(\xi-x)^2/(2\sigma^2\tau)}\, d\xi$$

where $f \in L_2(\mathbb{R})$, and note that

$$\begin{aligned} \|F(\cdot,\tau)\|_{L_2((I_1^*)^c)} &= \Big(\int_{(I_1^*)^c} \Big| \int_{-\infty}^{\infty} f(\xi)(\xi-x)e^{-(\xi-x)^2/(2\sigma^2\tau)}\, d\xi \Big|^2 dx\Big)^{1/2} \\ &\le A_1(\tau) + A_2(\tau) \end{aligned}$$

where

$$\begin{aligned} A_1(\tau) &:= \Big(\int_{(I_1^*)^c} \Big| \int_{x^-/2}^{x^+/2} f(\xi)(\xi-x)e^{-(\xi-x)^2/(2\sigma^2\tau)}\, d\xi \Big|^2 dx\Big)^{1/2}, \\ A_2(\tau) &:= \Big(\int_{(I_1^*)^c} \Big| \int_{\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2})} f(\xi)(\xi-x)e^{-(\xi-x)^2/(2\sigma^2\tau)}\, d\xi \Big|^2 dx\Big)^{1/2}. \end{aligned}$$

We bound the integrals $A_1(\tau)$, $A_2(\tau)$ using the Cauchy-Buniakovski-Schwarz inequality. First,

$$\begin{aligned} A_1(\tau) &\le \|f\|_{L_2(\frac{x^-}{2},\frac{x^+}{2})} \Big(\int_{(I_1^*)^c} \Big(\int_{\frac{x^-}{2}}^{\frac{x^+}{2}} (\xi-x)^2 e^{-(\xi-x)^2/(\sigma^2\tau)}\, d\xi\Big) dx\Big)^{1/2} \\ &\le \tau^{3/4}\|f\|_{L_2(\mathbb{R})} \Big(\int_{(I_1^*)^c} \Big(\int_{(x-\frac{x^-}{2})/\sqrt{\tau}}^{(x-\frac{x^+}{2})/\sqrt{\tau}} v^2 e^{-v^2/\sigma^2}\, dv\Big) dx\Big)^{1/2}. \end{aligned}$$

The integration endpoints $(x-\frac{x^-}{2})/\sqrt{\tau}$ and $(x-\frac{x^+}{2})/\sqrt{\tau}$ tend to $\pm\infty$. Applying the asymptotic Laplace method we have

$$\begin{aligned} A_1(\tau) &\le C_1\tau^{3/4}\|f\|_{L_2(\mathbb{R})} \Big(\int_{(I_1^*)^c} \Big(\frac{|x-\frac{x^+}{2}|}{\sqrt{\tau}} e^{-(x-x^+/2)^2/(\sigma^2\tau)} \\ &\quad + \frac{|x-\frac{x^-}{2}|}{\sqrt{\tau}} e^{-(x-x^-/2)^2/(\sigma^2\tau)}\Big) dx\Big)^{1/2} \\ &\le C_2\tau^{1/2}\|f\|_{L_2(\mathbb{R})} \Big(\frac{|x^+-1|}{2} e^{-((x^+-1)/2)^2/(2\sigma^2\tau)} \\ &\quad + \frac{|x^-+1|}{2} e^{-((x^-+1)/2)^2/(2\sigma^2\tau)}\Big) \end{aligned}$$

where C_1, C_2 are constants that do not depend on x and τ. Analogously,

$$\begin{aligned} A_2(\tau) &\le \|f\|_{L_2(\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2}))}\left(\int_{(I_1^*)^c}\left(\int_{\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2})}(\xi-x)^2 e^{-(\xi-x)^2/(\sigma^2\tau)}d\xi\right)dx\right)^{1/2} \\ &\le \tau^{3/4}\|f\|_{L_2(\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2}))}\left(\int_{(I_1^*)^c}\left(\int_{-\infty}^{\infty} v^2 e^{-v^2/\sigma^2}\,dv\right)dx\right)^{1/2} \\ &= \sqrt{2}\,\tau^{3/4} d(\sigma)\|f\|_{L_2(\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2}))}, \end{aligned}$$

where

$$d(\sigma) = \Big(\int_{-\infty}^{\infty} v^2 e^{-v^2/\sigma^2}\,dv\Big)^{1/2}.$$

These bounds on $A_1(\tau)$, $A_2(\tau)$ give

$$\begin{aligned} \|F(\cdot,\tau)\|_{L_2((I_1^*)^c)} &\le C\Big[\tau^{1/2}\|f\|_{L_2(\mathbb{R})}\Big(\frac{|x^+|}{2}e^{-((x^+-1)/2)^2/(2\sigma^2\tau)} \\ &\qquad + \frac{|x^-|}{2}e^{-((x^-+1)/2)^2/(2\sigma^2\tau)}\Big) \\ &\qquad + \tau^{3/4}\|f\|_{L_2(\mathbb{R}\setminus(x^-/2,x^+/2))}\Big] \end{aligned} \tag{4.14}$$

where the constant C does not depend on x and τ. Now recalling (4.13) we apply (4.14) to $f(\xi) = g(\xi-\mu t-\sigma_j)(\xi-x)$ for each j to obtain the following estimate: if we denote

$$a(x^-,x^+,\tau) := \Big(\frac{|x^+|}{2}e^{-(x^+-1/2)^2/(2\sigma^2\tau)} + \frac{|x^-|}{2}e^{-((x^-+1)/2)^2/(2\sigma^2\tau)}\Big)$$

then

$$\begin{aligned} \|\frac{\partial}{\partial x}u_\infty(x,\tau)\|_{L_2((I_1^*)^c)} &\le \frac{c}{\sigma^3\tau\sqrt{2\pi}}e^{-rt}\Big(\sum_{j=-\infty}^{\infty}|c_j(\tau)|\Big)\|g\|_{L_2(\mathbb{R})}a(x^-,x^+,\tau) \\ &\quad + \frac{c}{\sigma^3\tau^{3/4}\sqrt{2\pi}}\sum_{j=-\infty}^{\infty}|c_j(\tau)|\cdot\|g(\cdot+(\mu+\sigma_j))\|_{L_2(\mathbb{R}\setminus(\frac{x^-}{2},\frac{x^+}{2}))}. \end{aligned}$$

It is easy to show that the convergence of the series $\sum_{j=-\infty}^{\infty}|c_j(\tau)|$ is uniform in $\tau\in[0,t]$. So on the one hand

$$\sum_{j=-\infty}^{\infty}|c_j(\tau)| \le M,$$

where M does not depend on τ, and on the other hand for any $\epsilon > 0$ there exists $N = N(\epsilon)$ not depending on τ such that

$$\sum_{|j|>N}|c_j(\tau)| \le \epsilon.$$

Thus we have that

$$\begin{aligned} \|\frac{\partial}{\partial x}u_\infty(x,\tau)\|_{L_2((I_1^*)^c)} &\le \frac{cM}{\sigma^3\tau\sqrt{2\pi}}\|g\|_{L_2(\mathbb{R})}e^{-r\tau}a(x^-,x^+,\tau) \\ &\quad + \frac{cM}{\sigma^2\tau^{3/4}\sqrt{2\pi}}e^{-rt}\|g\|_{L_2(\mathbb{R}\setminus(\frac{x^-}{4},\frac{x^+}{4}))} \\ &\quad + \frac{c}{\sigma^3\tau^{3/4}\sqrt{2\pi}}e^{-rt}\|g\|_{L_2(\mathbb{R})}\cdot\epsilon. \end{aligned} \tag{4.15}$$

Here we have taken $|x^{\pm}|$ large enough such that for $|j| \leq N$ and $x \in \mathbb{R} \setminus (x^-/2, x^+/2)$,

$$x + (\mu t + \sigma_j) \in \mathbb{R} \setminus (\frac{x^-}{4}, \frac{x^+}{4}).$$

Therefore (4.15) implies the following estimate,

$$\|\frac{\partial}{\partial x} u_\infty(x,\tau)\|_{L_2((I_1^*)^c)} \leq C_0\Big(\frac{a(x^-,x^+,\tau)}{\tau} + \frac{b(x^*)}{\tau^{3/4}}\Big),$$

where $x^* = \min(|x^-|, |x^+|)$, C_0 is a constant not depending on τ, and as required, we have $\lim_{x\to\infty} b(x) = 0$. This concludes the proof. □

The analysis of Theorem 4.4 allows us to be more precise regarding the rate of convergence when we impose additional conditions on $g(x)$ and $J_0(\xi)$. In particular the following result holds.

Theorem 4.5. *Let g be a function in $L_2(\mathbb{R})$ and of compact support. If the almost-periodic part $J_0(\xi)$ of the characteristic function of Q is periodic and has bounded k-th derivative, $k \geq 2$, then for $|x^{\pm}|$ large enough,*

$$\|\frac{\partial}{\partial x} u_\infty(x,\tau)\|_{L_2((I_1^*)^c)} \leq C_k\left(\frac{e^{-(x^*/2)^2/(2\sigma^2\tau)}}{x^*\tau} + \frac{1}{(x^*)^{k-1}\tau^{3/4}}\right)$$

where C_k does not depend on τ.

If moreover, the function $J_0(\xi)$ admits analytic continuation in a horizontal strip $\{\xi : |\mathrm{Im}\xi| \leq \rho\}$ such that condition (4.11) holds, then

$$\|\frac{\partial}{\partial x} u_\infty(x,\tau)\|_{L_2((I_1^*)^c)} \leq C_{\tilde\rho}\left(\frac{e^{-(x^*/2)^2/(2\sigma^2\tau)}}{x^*\tau} + \frac{e^{-\tilde\rho x^*}}{\tau^{3/4}}\right)$$

where $\tilde\rho < \rho$ is any positive number and $C_{\tilde\rho}$ does not depend on τ.

5 Asymptotics of Barrier Options

We now return to the question of barriers. In this section we will determine the asymptotics of the option price as the barriers $x^{\pm}$ tend to $\pm\infty$ for Lévy processes with jumps.

5.1 Auxiliary heat equation

By construction, the function u_∞ defined by (4.1) or (4.2) solves the problem

$$\frac{\partial}{\partial t} u_\infty(x,t) + (A_\infty u_\infty)(x,t) = 0, \quad (x,t) \in \mathbb{R}\times(0,\infty), \tag{5.1}$$

$$u_\infty(x,0) = g(x), \tag{5.2}$$

where the operator A_∞ is defined analogously to (3.1) but without reference to barriers:

$$\begin{aligned}(A_\infty f)(x) &= -\frac{\sigma^2}{2} f''(x) - \mu f'(x) + rf(x) + \lambda f(x) - \lambda \sum_{j=-\infty}^{\infty} p_j f(x+y_j) \\ &= (Af)(x) + \lambda \sum_{j=-\infty}^{\infty} p_j f(x+y_j)\mathbf{1}_{(I^*)^c}(x+y_j) \end{aligned} \tag{5.3}$$

for $x \in \mathbb{R}$.

We think of A_∞ as an operator on $L_2(\mathbb{R})$ with the (dense) domain

$$\mathcal{D}(A_\infty) = C_2^2(\mathbb{R}) \cap L_2(\mathbb{R}),$$

where $C_2^2(\mathbb{R})$ is the set of the all functions having continuous first and second derivatives belonging to $L_2(\mathbb{R})$.

Recall the set I_1^* introduced in Subsection 4.3. Let $\chi \in C^\infty(\mathbb{R})$ have support $\operatorname{supp}\chi \subseteq I^*$ and satisfy $\chi(x) = 1$ for all x in a neighborhood of $\operatorname{clos} I_1^*$, so in particular $\chi^{(k)}(x^- + 1) = \chi^{(k)}(x^+ - 1) = 0$ for $k \geq 1$. We introduce the difference

$$u_0(x,t) := u_\infty(x,t)\chi(x) - u(x,t), \quad x \in I^*, \tag{5.4}$$

where $u(x,t)$ is the solution of problem (3.5)–(3.7). Since the application of barriers tends to reduce the value of an option, one would naturally expect that $u_0(x,t) > 0$.

Let us evaluate the corresponding heat equation operator for u_0. By (3.2) and (5.3),

$$\frac{\partial}{\partial t}u_0 + Au_0 = (\frac{\partial}{\partial t}u_\infty)\chi + A(u_\infty\chi);$$

note that the summation $\sum p_j u_\infty(x+y_j,t)\chi(x+y_j)\mathbf{1}_{(I^*)^c}(x+y_j)$ has vanished, since $\chi\mathbf{1}_{(I^*)^c} = 0$ identically. We expand the last term to obtain

$$\begin{aligned}(A(u_\infty\chi))(x,t) &= -\Big(\frac{\sigma^2}{2}\frac{\partial^2 u_\infty}{\partial x^2} + \mu\frac{\partial u_\infty}{\partial x} + (r-\lambda)u_\infty\Big)(x,t)\chi(x)\\ &\quad -u_\infty(x,t)\Big(\frac{\sigma^2}{2}\chi''(x) + \mu\chi'(x)\Big)\\ &\quad -\lambda\sum p_j u_\infty(x+y_j,t)\chi(x+y_j).\end{aligned}$$

In the light of this expression, since we consider problems (3.2)–(3.3) and (5.1)–(5.2) with the same $g(x)$, our auxiliary function $u_0(x,t)$ solves by construction the problem

$$\begin{aligned}\frac{\partial}{\partial t}u_0(x,t) + Au_0(x,t) &= f_1(x,t) + f_2(x,t) + f_3(x,t),\\ &\qquad (x,t) \in I^* \times (0,\infty), \qquad (5.5)\\ u_0(x,0) &= g_1(x), \quad x \in I^*, \qquad (5.6)\\ u_0(x^-,t) &= u_0(x^+,t) = 0, \quad t \in (0,\infty), \qquad (5.7)\end{aligned}$$

where we have abbreviated

$$\begin{aligned}f_1(x,t) &= \lambda\sum_{j=-\infty}^{\infty} p_j u_\infty(x+y_j,t)(\chi(x+y_j) - \chi(x)),\\ f_2(x,t) &= -u_\infty(x,t)\Big(\frac{\sigma^2}{2}\chi''(x) + \mu\chi'(x)\Big), \text{ and}\\ f_3(x,t) &= -\sigma^2\frac{\partial u_\infty(x,t)}{\partial x}\chi'(x),\\ g_1(x) &= g(x)(\chi(x) - 1).\end{aligned}$$

5.2 Asymptotics of auxiliary function

We will show that f_1, f_2, f_3, and g_1 have L_2-norms tending to zero when $x^\pm \to \infty$. It will follow that the (unique) solution of problem (5.5)–(5.7) is small, see (3.10)–(3.11). More exactly, we have the following result.

Theorem 5.1. *If* $g \in L_2(\mathbb{R})$*, then the problem* (5.5)–(5.7) *has a unique solution* u_0*. This solution satisfies*

$$\lim_{x^\pm \to \pm\infty} \|u_0(x,t)\|_{L_2(I^*)} = 0.$$

Proof. According to (3.11) and (5.5),

$$\|u_0(\cdot,t)\|_2 \le e^{-rt}\|g_1\|_2 + \int_0^t (\|f_1(\cdot,\tau)\|_2 + \|f_2(\cdot,\tau)\|_2 + \|f_3(\cdot,\tau)\|_2)\, d\tau. \tag{5.8}$$

From the definition of g_1 it follows easily that $\lim_{x^\pm \to \pm\infty} \|g_1\|_2 = 0$. We proceed to estimate $f_1(x,t)$ as follows,

$$\|f_1(\cdot,t)\| \le 2\lambda \sum_{j=-\infty}^{\infty} p_j \|u_\infty(\cdot,t)\|_{L_2(I^*(y_j))},$$

where we have taken the norms over the shifted intervals

$$I^*(y_j) := \begin{cases} (x^- - y_j,\ x^- + 1) \cup (x^+ - y_j,\ x^+), & y_j > 0, \\ (x^-,\ x^- + y_j) \cup (x^+ - 1,\ x^+ - y_j), & y_j < 0. \end{cases}$$

Let $\epsilon > 0$ be small. Then take $N = N(\epsilon)$ large enough so that

$$2\lambda \|u_\infty(\cdot,t)\|_2 \sum_{|j|>N} p_j \le \epsilon.$$

Then we have

$$\|f_1(\cdot,t)\|_2 \le \lambda \sum_{j=-N}^{N} p_j \|u_\infty(\cdot,t)\|_{L_2(I^*(y_j))} + \epsilon.$$

According to Theorem 4.1, each term above of the sum tends to zero uniformly in t when $x^\pm \to \pm\infty$, so for $|x^\pm|$ large enough we have

$$\|f_1(\cdot,t)\|_2 \le \epsilon + \epsilon = 2\epsilon$$

and thus $\lim_{x^\pm \to \pm\infty} \|f_1(\cdot,t)\|_2 = 0$.

Next, taking into in account that the function $(\sigma^2/2)\chi''(x) + \mu\chi'(x)$ is bounded with support in $(I_1^*)^c$ we have $\lim_{x^\pm \to \pm\infty} \|f_2(\cdot,\tau)\|_2 = 0$. Note that the convergence is uniform in $\tau \in [0,t]$.

Finally, we use Theorem 4.4 to estimate the last term in (5.8). Noting that $\chi'(x)$ is bounded with support in $(I_1^*)^c$, say $|\chi'(x)| < M'$ we have

$$\begin{aligned} \int_0^t \|f_3(\cdot,\tau)\|_2 d\tau &\le \sigma^2 M' \int_0^t \|\frac{\partial u}{\partial x}(\cdot,\tau)\|_{L_2((I_1^*)^c)}\, d\tau \\ &\le \sigma^2 C_0 M' \Big(\frac{1}{x^*} \int_0^t \frac{1}{\tau} e^{-(x^*/2)^2/(2\sigma^2\tau)}\, d\tau + b(x^*) \int_0^t \frac{d\tau}{\tau^{3/4}} \Big) \\ &\le \sigma^2 C_0 M' \Big(\frac{1}{x^*} \int_0^{8t\sigma^2/(x^*)^2} \frac{1}{v} e^{-1/v}\, dv + 4b(x^*) t^{1/4} \Big) \end{aligned}$$

where $x^* := \min(|x^-|, |x^+|)$ and C_0, $b(x)$ are as in Theorem 4.4. This clearly tends to zero as $x^* \to \infty$, because $(1/v)e^{-1/v}$ is integrable and the interval of integration degenerates. Thus the proof is complete. □

Theorem 5.1 justifies that $u_\infty(x,t)$ is the main term of the asymptotic expansion of $u(x,t)$. Indeed, we have shown

$$u(x,t) = u_\infty(x,t) + O(x,t) \tag{5.9}$$

in the L_2-sense, where $O(x,t)$ refers to (see (5.4))

$$-u_\infty(x,t)(1-\chi(x)) - u_0(x,t).$$

Thus according to Theorem 4.1, for arbitrary $g \in L_2(\mathbb{R})$ and characteristic exponent $\psi^Q(\xi)$ of (2.1) we have that

$$\lim_{x^\pm \to \pm\infty} \|O(x,t)\|_2 = 0. \tag{5.10}$$

The rate of convergence in (5.10) depends on properties of the functions $g(x)$ and $\psi^Q(\xi)$. Suppose that $\operatorname{supp} g(x)$ is compact and that the function $J_0(\xi)$ has bounded k-th derivative, for some $k \geq 2$. Then from Theorems 4.3 and 4.5,

$$\|O(\cdot,t)\|_2 \leq d_1(k)\,(x^*)^{-(k-1)},$$

where $d_1(k)$ is a constant that does not depend on x^- and x^+. If furthermore $J_0(\xi)$ admits analytic continuation in a horizontal strip $\{|\operatorname{Im}\xi| < \rho\}$ such that condition (4.12) holds, then

$$\|O(\cdot,t)\|_2 \leq d_2(\rho)e^{-d_3 x^*} \tag{5.11}$$

where the positive constants $d_2(\rho)$, d_3 do not depend on x^- and x^+.

6 Asymptotics: Black-Scholes Case

We now consider the situation that $J_0(\xi) \equiv 0$; that is, the characteristic exponent in (2.1) (see (2.9)) with respect to the (unique) equivalent martingale measure can be reduced to the form

$$\psi^Q(\xi) = \frac{\sigma^2}{2}\xi^2 - i(r - \frac{\sigma^2}{2})\xi. \tag{6.1}$$

This is the classical Black-Scholes model. The results of Section 5 can be specialized to this case, and from (5.9)–(5.10) we see that $u_\infty(x,t)$ is the main term of the asymptotic expansion for the function $u(x,t)$ in the sense of the L_2-norm as $x^\pm \to \pm\infty$. Moreover, if $\operatorname{supp} g(x)$ is compact, then by (5.11) the remainder term $O(\cdot,t)$ in (5.9) has the following norm estimate,

$$\|O(\cdot,t)\|_2 \leq d_\rho\, e^{-\rho x^*}, \tag{6.2}$$

where d_ρ depends only on the half-width ρ of the band on which $J_0(\xi)$ is analytic, and not on t. We apply a variant of the *method of images* (see for example [17, 20, 21, 23]), considered from the point of view of asymptotics.

6.1 Exact solution for Black-Scholes equation

In the Black-Scholes context we may obtain more precise results since problem (3.2)–(3.4) has an exact solution. We shall produce the complete asymptotic expansion.

Consider the problem (3.5)–(3.7). Since $J_0(\xi) \equiv 0$ we have that (3.5) reduces to

$$u_t(x,t) = \frac{\sigma^2}{2} u_{xx}(x,t) - (r - \frac{\sigma^2}{2}) u_x(x,t) - ru(x,t),$$

for $(x,t) \in I^* \times (0,\infty)$. Introduce the function $U(x,t)$ defined by

$$u(x,t) =: e^{\alpha x + \beta t} U(x,t), \tag{6.3}$$

where

$$\alpha := -\frac{1}{2}(1 - \frac{2r}{\sigma^2}), \quad \beta := -\frac{\sigma^2}{8}(1 + \frac{2r}{\sigma^2})^2. \tag{6.4}$$

This function satisfies the following heat equation:

$$\begin{aligned} U_t(x,t) &= \frac{\sigma^2}{2} U_{xx}(x,t), \quad (x,t) \in I^* \times (0,\infty), & (6.5)\\ U(x,0) &= g_\alpha(x), \quad x \in I^*, & (6.6)\\ U(x^-,t) &= U(x^+,t) = 0, \quad t \in (0,\infty), & (6.7) \end{aligned}$$

where

$$g_\alpha(x) := g(x) e^{-\alpha x}. \tag{6.8}$$

In analogy to (6.3), let us define $U_\infty(x,t)$ via $u_\infty(x,t) = e^{\alpha x + \beta t} U_\infty(x,t)$. Observe that our assumption $J_0(\xi) \equiv 0$ amounts to setting $\lambda = 0$ in (5.3). From this it follows that U_∞ is a solution to problem (6.5)–(6.7) for all $x \in \mathbb{R}$ (rather than just $x \in I^*$), i.e., without barriers.

It is easy to see from (4.2) that

$$U_\infty(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t\frac{\sigma^2}{2}\xi^2 + i\xi x} \widehat{g_\alpha}(\xi)\, d\xi. \tag{6.9}$$

In analogy with (5.4) we introduce the difference

$$U_0(x,t) := U_\infty(x,t) - U(x,t),$$

which solves the following problem:

$$\begin{aligned} U_{0,t}(x,t) &= \frac{\sigma^2}{2} U_{0,xx}(x,t), \quad (x,t) \in I^* \times (0,\infty), & (6.10)\\ U_0(x,0) &= 0, \quad x \in I^*, & (6.11)\\ U_0(x^-,t) &= U_\infty(x^-,t), \quad t \in (0,\infty), & (6.12)\\ U_0(x^+,t) &= U_\infty(x^+,t), \quad t \in (0,\infty). & (6.13) \end{aligned}$$

We can write down the exact solution to problem (6.10)–(6.13) by using the Laplace transform V_0 of U_0,

$$V_0(x,\omega) := (\mathcal{L}U_0)(x,\omega) = \int_0^\infty U_0(x,t) e^{-\omega t} dt, \quad \omega \in \mathbb{C}.$$

Applying this transform to (6.10), (6.12)–(6.13) and taking into account (6.11) we find

$$\begin{aligned}\frac{\sigma^2}{2}V_{0,xx}(x,\omega)-\omega V_0(x,\omega) &= 0, \quad x\in I^* && (6.14)\\ V_0(x^+,\omega) &= \epsilon^+(\omega), && (6.15)\\ V_0(x^-,\omega) &= \epsilon^-(\omega), && (6.16)\end{aligned}$$

where

$$\begin{aligned}\epsilon^{\pm}(\omega) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ivx^{\pm}}\widehat{g_\alpha}(v)}{(\frac{\sigma^2}{2}v^2+\omega)}\,dv\\ &= \int_{-\infty}^{\infty}\gamma(x^{\pm}-\xi,\omega)g_\alpha(\xi)\,d\xi\end{aligned}$$

in which we recall that g_α was defined in (6.8) and we write for simplicity

$$\gamma(\xi,\omega) := \frac{\sigma'}{2\sqrt{\omega}}e^{-\sigma'\sqrt{\omega}|\xi|}, \quad \sigma' := \sqrt{2}/\sigma, \tag{6.17}$$

taking the branch of $\sqrt{\omega}$ such that $\mathrm{Re}\,\sqrt{\omega}\geq 0$. The problem (6.14)–(6.16) is an ordinary differential equation with constant coefficients and its (unique) continuous solution is

$$\begin{aligned}V_0(x,\omega) &= \frac{\left(\epsilon^+(\omega)-\epsilon^-(\omega)e^{-\sigma'\sqrt{\omega}(x^+-x^-)}\right)e^{-\sigma'\sqrt{\omega}(x-x^-)}}{\left(1-e^{-2\sigma'\sqrt{\omega}(x^+-x^-)}\right)}\\ &\quad+\frac{\left(\epsilon^-(\omega)-\epsilon^+(\omega)e^{-\sigma'\sqrt{\omega}(x^+-x^-)}\right)e^{\sigma'\sqrt{\omega}(x-x^+)}}{\left(1-e^{-2\sigma'\sqrt{\omega}(x^+-x^-)}\right)} \qquad (6.18)\end{aligned}$$

for $x\in I^*$. From this we recover U_0 via the inverse Laplace transform,

$$U_0(x,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}V_0(x,\omega)e^{\omega t}\,d\omega,$$

which we can express by means of the geometric series

$$\frac{1}{1-e^{-2\sigma'\sqrt{\omega}(x^+-x^-)}}=\sum_{k=0}^{\infty}e^{-2k\sigma'\sqrt{\omega}(x^+-x^-)}, \quad \omega\neq 0,$$

in the form

$$\begin{aligned}U_0(x,t) &= \sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big(\epsilon^+(\omega)e^{-R^+_{2k}(x)\sigma'\sqrt{\omega}}+\epsilon^-(\omega)e^{-R^-_{2k}(x)\sigma'\sqrt{\omega}}\\ &\quad -\epsilon^-(\omega)e^{-R^+_{2k+1}(x)\sigma'\sqrt{\omega}}-\epsilon^+(\omega)e^{-R^-_{2k+1}(x)\sigma'\sqrt{\omega}}\Big)e^{t\omega}\,d\omega \qquad (6.19)\end{aligned}$$

for $x\in I^*$; this formula contains the abbreviations

$$\begin{aligned}R^+_k(x) &:= (x-x^-)+k(x^+-x^-),\\ R^-_k(x) &:= (x^+-x)+k(x^+-x^-) && (6.20)\end{aligned}$$

for $k = 0, 1, \dots$ In order to calculate (6.19) we must evaluate the integrals

$$\begin{aligned} A^{\pm}_{\pm,k} &:= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \epsilon^{\pm}(\omega) e^{-R^{\pm}_{k}(x)\sigma' \sqrt{\omega}} e^{\omega t}\, d\omega \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Big(\int_{-\infty}^{\infty} \gamma(\xi - x^{\pm}, \omega) g_{\alpha}(\xi)\, d\xi \Big) e^{-R^{\pm}_{k}(x)\sigma' \sqrt{\omega}} e^{\omega t}\, d\omega \\ &= \frac{\sigma'}{2\pi i} \int_{-\infty}^{\infty} g_{\alpha}(\xi) \int_{-i\infty}^{i\infty} \frac{e^{-R^{\pm}_{k}(x)\sigma' \sqrt{\omega} - \sigma' |\xi - x^{\pm}| \sqrt{\omega} + \omega t}}{2\sqrt{\omega}}\, d\omega\, d\xi \end{aligned}$$

In these integrals, the lower index $\pm$ of $A^{\pm}_{\pm,k}$ is understood to agree with the sign in $\epsilon^{\pm}(\omega)$, the upper with $R^{\pm}_{k}(x)$. Recall that $\gamma(\xi, \omega)$ is defined in (6.17). Let us write also

$$R^{\pm}_{\pm,k}(x, \xi) := R^{\pm}_{k}(x) + |\xi - x^{\pm}|$$

with all four combinations of signs, and the lower index $\pm$ referring to the choice of $x^{\pm}$. With this notation we arrive at

$$A^{\pm}_{\pm,k} = \frac{\sigma'}{4\pi i} \int_{-\infty}^{\infty} g_{\alpha}(\xi) \Big(\int_{-i\infty}^{i\infty} \frac{e^{-\sigma' R^{\pm}_{\pm,k}(x,\xi) \sqrt{\omega} + \omega t}}{\sqrt{\omega}}\, d\omega \Big)\, d\xi. \tag{6.21}$$

We now concentrate on the inner integral in (6.21). For the moment, we fix k and the choice of signs, as well as the values of x, t, and ξ. With this value of $R^{\pm}_{\pm}(x, \xi)$, the integrand is an analytic function of ω in the complement of the semiaxis $\{\omega < 0\}$, and thus we can deform the contour of integration $(-i\infty, i\infty)$ to the double contour $\Gamma = \Gamma_{+} \cup \Gamma_{-}$ where

$$\begin{aligned} \Gamma_{+} &= \Big\{ \omega = \Big(\frac{\sigma' R^{\pm}_{\pm,k}(x,\xi)}{2t} + \frac{iy}{\sqrt{t}} \Big)^{2},\ y \in (0, \infty) \Big\}, \\ \Gamma_{-} &= \Big\{ \omega = \Big(\frac{\sigma' R^{\pm}_{\pm,k}(x,\xi)}{2t} - \frac{iy}{\sqrt{t}} \Big)^{2},\ y \in (0, \infty) \Big\}. \end{aligned}$$

Via the changes of variable $\omega = (\sigma' R^{\pm}_{\pm,k}(x,\xi)/(2t) \pm iy/\sqrt{t})^{2}$ we obtain

$$\begin{aligned} \int_{-i\infty}^{i\infty} \frac{e^{-\sigma' R^{\pm}_{\pm,k}(x,\xi) \sqrt{\omega} + \omega t}}{\sqrt{\omega}}\, d\omega &= \frac{4i}{\sqrt{t}} \int_{0}^{\infty} e^{-\frac{\sigma'^{2}(R^{\pm}_{\pm,k}(x,\xi))^{2}}{4t} - y^{2}}\, dy \\ &= 4i \sqrt{\frac{2\pi}{t}} \exp\{ \frac{-\sigma'^{2}(R^{\pm}_{\pm,k}(x,\xi))^{2}}{4t} \}. \end{aligned}$$

Thus (6.21) becomes

$$A^{\pm}_{\pm,k} = \frac{2\sigma'}{\sqrt{2\pi t}}\, e^{\frac{-\sigma'^{2}(R^{\pm}_{k}(x))^{2}}{4t}}\, B^{\pm}_{\pm,k}(x)$$

where we denote

$$B^{\pm}_{\pm,k}(x) := \int_{-\infty}^{\infty} g_{\alpha}(\xi) e^{-(1/(2\sigma^{2} t))(2|\xi - x^{\pm}| R^{\pm}_{k}(x) + |\xi - x^{\pm}|^{2})}\, d\xi. \tag{6.22}$$

In this notation the two appearances of $x^\pm$ share the same sign, which matches the lower index of $B^\pm_{\pm,k}(x)$, whose upper index matches that of $R^\pm_k(x)$. With this, (6.19) can be rewritten as

$$\begin{aligned} U_0(x,t) &= \frac{2\sigma'}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} \Big(e^{-(1/(2\sigma^2 t))(R^+_{2k}(x))^2} B^+_{+,2k}(x) \\ &\qquad + e^{-(1/(2\sigma^2 t))(R^-_{2k}(x))^2} B^-_{-,2k}(x) \\ &\qquad - e^{-(1/(2\sigma^2 t))(R^-_{2k+1}(x))^2} B^+_{-,2k+1}(x) \\ &\qquad - e^{-(1/(2\sigma^2 t))(R^+_{2k+1}(x))^2} B^-_{+,2k+1}(x) \Big) \end{aligned} \tag{6.23}$$

for $x \in I^*$. Taking into account the definition (6.3) we see that we have arrived at the following result.

Theorem 6.1. *The fair value $u(x,t)$ of the double-barrier option in the market model with no jump discontinuities is equal to*

$$u(x,t) = u_\infty(x,t) - u_0(x,t), \tag{6.24}$$

where $u_\infty(x,t) = e^{\alpha x + \beta t} U_\infty(x,t)$ is the price of a European option (i.e. without barriers) as given by (6.9)*, and $u_0(x,t) = e^{\alpha x+\beta t} U_0(x,t)$ with $U_0(x,t)$ given by* (6.23)*.*

Since we are working with a model with no jumps, the exact form of the solution $u_\infty(x,t)$ can also be obtained directly from (4.5) by setting $J_0(\xi) \equiv 0$:

$$u_\infty(x,t) = \frac{e^{-rt}}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} g(\xi + t(r - \frac{\sigma^2}{2}))\, e^{-(\xi - x)^2/(2\sigma^2 t)}\, d\xi. \tag{6.25}$$

Formula (6.24) is the exact solution of problem (6.3),(3.6),(3.7).

6.2 Asymptotic formula for barrier option in Black-Scholes model

We will now see that the expression (6.24) with $U_0(x,t)$ given by the series (6.23) is an asymptotic expansion of the function $u(x,t)$ as $x^\pm \to \pm\infty$. More precisely, the following results hold. Recall the functions $R^\pm_k(x)$ in our analysis of U_0.

Theorem 6.2. *Let $g \in L_2(\mathbb{R})$, and let x be a fixed element of I^*. For fixed k and fixed signs $\pm$, the value* (6.22) *satisfies*

$$B^\pm_{\pm,k}(x) = O\Big(\frac{1}{(R^\pm_k(x))^{1/2}}\Big) \tag{6.26}$$

as $|x^\pm|$ tend to infinity. If $g \in L_2(\mathbb{R}) \cap C^0_0(\mathbb{R})$, where $C^0_0(\mathbb{R})$ is the set of all continuous functions on $\mathbb{R}$ vanishing at infinity, then (6.22) *satisfies*

$$B^\pm_{\pm,k}(x) = 2\sigma^2 t e^{-\alpha x^\pm} \frac{g(x^\pm)}{R^\pm_k(x)} \Big(1 + O\Big(\frac{1}{R^\pm_k(x)}\Big)\Big). \tag{6.27}$$

If supp $g \subseteq [a^-, a^+]$ *and* $g \in C^0[a^-, a^+]$ *then*

$$\begin{aligned} B^{\pm}_{\pm,k}(x) &= 2\sigma^2 e^{-\alpha x^{\pm}} \frac{g(x^{\pm})}{R^{\pm}_k(x)} \exp\Big\{-\frac{1}{2\sigma^2 t}(2|a^{\pm} - x^{\pm}|R^{\pm}_k(x) + |a^{\pm} - x^{\pm}|^2 \\ &\quad -\alpha|a^{\pm} - x^{\pm}|)\Big\} \cdot \Big(1 + O\Big(\frac{1}{R^{\pm}_k(x)}\Big)\Big) \end{aligned} \tag{6.28}$$

where the signs of $a^{\pm}$, $x^{\pm}$ *agree with the lower index of* $B^{\pm}_{\pm,k}(x)$.

Note that when $x^{\pm} \to \pm\infty$ in such a way that the proportion $x^+/(-x^-)$ is bounded above and below, one may replace $O(1/R^{\pm}_k(x))$ with $O(1/x^+)$ in these asymptotic expansions.

Proof. From (6.22),

$$\begin{aligned} |B^{\pm}_{\pm,k}(x)| &\le \Big|\int_{|\xi - x^{\pm}| < \frac{|x^{\pm}|}{2}} (\cdot)\, d\xi\Big| + \Big|\int_{|\xi - x^{\pm}| > \frac{|x^{\pm}|}{2}} (\cdot)\, d\xi\Big| \\ &\le \Big(\int_{|\xi - x^{\pm}| < \frac{|x^{\pm}|}{2}} |g(\xi)|^2\, d\xi\Big)^{1/2} \\ &\quad \cdot\Big(\int_{|\xi - x^{\pm}| \le \frac{|x^{\pm}|}{2}} e^{-\frac{\sigma'^2}{2t}\left(2|\xi - x^{\pm}|R^{\pm}_k(x) + |\xi - x^{\pm}|^2\right) - 2\alpha\xi}\, d\xi\Big)^{1/2} \\ &\quad + \Big(\int_{|\xi - x^{\pm}| > \frac{|x^{\pm}|}{2}} |g(\xi)|^2\, d\xi\Big)^{1/2} \\ &\quad \cdot\Big(\int_{|\xi - x^{\pm}| \ge \frac{|x^{\pm}|}{2}} e^{-\frac{\sigma'^2}{2t}\left(2|\xi - x^{\pm}|R^{\pm}_k(x) + |\xi - x^{\pm}|^2\right) - 2\alpha\xi}\, d\xi\Big)^{1/2} \\ &\le \|g\|_{L_2(\frac{x^{\pm}}{2}, \frac{3x^{\pm}}{2})}\Big(2e^{\alpha x^{\pm}} \int_0^{\infty} e^{-\frac{\sigma'^2}{2t}(2|\xi'|R^{\pm}_k(x) + |\xi'|^2) - 2\alpha\xi'}\, d\xi'\Big)^{1/2} \\ &\quad + \|g\|_{L_2(\mathbb{R})}\Big(2e^{\alpha x^{\pm}} \int_{\frac{|x^{\pm}|}{2}}^{\infty} e^{-\frac{\sigma'^2}{2t}(2|\xi'|R^{\pm}_k(x) + |\xi'|^2) - 2\alpha\xi'}\, d\xi'\Big)^{1/2}. \end{aligned}$$

When we take into account that

$$\lim_{x^* \to \infty} \|g\|_{L_2(\frac{x^{\pm}}{2}, \frac{3x^{\pm}}{2})} = 0$$

and note that according to the asymptotic theory (see for example the *standard Laplace method* in [13] or [31])

$$\int_a^{\infty} e^{-\frac{\sigma'^2}{2t}(2|\xi'|R^{\pm}_k(x) + |\xi'|^2) - 2\alpha\xi'}\, d\xi' = \frac{t}{\sigma'^2} \frac{e^{-\frac{\sigma'^2}{2t}(aR^{\pm}_k(x) + a^2) - 2\alpha a}}{R^{\pm}_k(x)} \Big(1 + O\big(\frac{1}{R^{\pm}_k(x)}\big)\Big)$$

for $a \ge 0$, we arrive at (6.26). f Analogously, formulas (6.27)–(6.28) are also obtained from the asymptotic Laplace method [31]. □

Formulas (6.22)–(6.24) provide the main result of this section. Let us write $u_0^{(N)}(x, t)$ for the approximation for $u_0(x,t)$ obtained by the first N terms of (6.23),

$$\begin{aligned} u_0^{(N)}(x,t) &= \frac{2e^{\alpha x+\beta t}}{\sigma\sqrt{\pi t}} \sum_{k=0}^{N-1} \Big(\\ & e^{-\frac{1}{2\sigma^2 t}(R_{2k}^+(x))^2} B_{+,2k}^+(x) - e^{-\frac{1}{2\sigma^2 t}(R_{2k+1}^+(x))^2} B_{+,2k+1}^-(x) \\ & + e^{-\frac{1}{2\sigma^2 t}(R_{2k}^-(x))^2} B_{-,2k}^-(x) - e^{-\frac{1}{2\sigma^2 t}(R_{2k+1}^-(x))^2} B_{-,2k+1}^+(x) \Big). \end{aligned} \tag{6.29}$$

The following result specifies the rate of convergence of the series (6.29) when used as an approximation of the function $u_0(x,t)$ introduced in equation (5.4).

Theorem 6.3. *Let $g \in L_2(\mathbb{R})$. Then the series* (6.23) *for $U_0(x,t)$ converges absolutely for every $t > 0$ and $x \in I^*$. Further, for $|x^\pm|$ sufficiently large,*

$$\begin{aligned} &|u(x,t) - u_\infty(x,t) - u_0^{(N)}(x,t)| \le \\ &\tilde{C} e^{\alpha x+\beta t} \Big(e^{-\frac{1}{2\sigma^2 t} R_{2N}^+(x)^2} |B_{+,2N}^+(x)| + e^{-\frac{1}{2\sigma^2 t} R_{2N}^-(x)^2} |B_{-,2N}^-(x)| \Big) \end{aligned} \tag{6.30}$$

where $\tilde{C}$ is a constant, α and β are given by (6.3)*, and $R_k^\pm(x)$ and $B_{\pm,k}^\pm(x)$ are given by* (6.20) *and* (6.22) *respectively.*

In particular, for $N = 0$ we have

$$\begin{aligned} &|u(x,t) - u_\infty(x,t)| \le \\ &\tilde{C} e^{\alpha x+\beta t} \Big(|B_{+,0}^+(x)| e^{-\frac{1}{2\sigma^2 t}|x-x^-|^2} + |B_{-,0}^-(x)| e^{-\frac{1}{2\sigma^2 t}|x^+-x|^2} \Big). \end{aligned} \tag{6.31}$$

Remark 6.4. The estimate (6.31) is stronger than (6.2). Moreover, the estimates (6.30)–(6.31) hold for fixed $x \in I^*$, while (6.2) is only an L_2-estimate.

7 Numerical Results

Concerning numerical aspects of the main theorems, we will limit the discussion to one example which facilitates comparison to an existing formula: a payoff function corresponding to a "supershare" type option, that is, $(s/K^-)\mathbf{1}_{[K^-,K^+]}$ where $s = S_t$ is the market value of the stock and $0 < K^- < K^+$. By the relation (2.3), the payoff is expressed in terms of the logarithmic independent variable $x = \log(s/S_0)$ as $g(x) = (S_0/K^-)e^x\mathbf{1}_{[\log K^-,\log K^+]}$. To calculate u_∞, the option value without barriers, we consider for simplicity a market with no jumps, effectively fixing $\lambda = 0$ in (2.9). For such a market the coefficients $c_j(t)$ vanish when $j \neq 0$, leaving only one term in the sum in (4.5). On the other hand, it is well known [38] that the classical supershare value of u_∞ can be calculated directly using the classical Black-Scholes apparatus,

$$u_\infty(x,t) = \frac{s}{K^-}\left(\frac{1}{2}\operatorname{Erf}(\frac{g^+}{\sqrt{2}}) - \frac{1}{2}\operatorname{Erf}(\frac{g^-}{\sqrt{2}})\right), \tag{7.1}$$

where one defines $g^\pm = (1/(\sigma\sqrt{t}))(\log(s/K^\pm) + (r+\sigma^2/2)t)$ and $\operatorname{Erf}(\zeta) = (2/\sqrt{\pi})\int_{-\infty}^{\zeta} e^{-\xi^2}\,d\xi$.

It is not difficult to work out an explicit formula for the double barrier supershare with barriers $X^{\pm} = S_0 e^{x^{\pm}}$, by a standard application of separation of variables (see for example [18, 19, 32, 38]). Take α and β as given by (6.4), write $\alpha' = 1 - \alpha$ and for $j = 1, 2, \ldots$, define the quantities β_j and $I_j = I_j(x^-, x^+)$ by $\beta_j = j\pi/(x^+ - x^-)$ and

$$\begin{aligned} I_j &= \frac{1}{\alpha'^2 + \beta_j^2}\Big(\alpha'(e^{\alpha' k^-}\sin(\beta_j(x^+ - k^-)) - e^{\alpha' k^+}\sin(\beta_j(x^+ - k^+)) \\ &\qquad - \beta_j(e^{\alpha' k^-}\cos(\beta_j(x^+ - k^+)) - e^{\alpha' k^-}\sin(\beta_j(x^+ - k^-))\Big) \end{aligned}$$

where $k^{\pm} = \log(K^{\pm}/S_0)$. Then the supershare option with barriers $x^{\pm}$ is given by

$$u(x,t) = \frac{2(s/K^-)^{\alpha} e^{(1/2)\beta\sigma^2 t}}{x^+ - x^-} \sum_{j=1}^{\infty} e^{-(1/2)\sigma^2 \beta_j^2 t} I_j \sin(\beta_j(x - x^+)). \tag{7.2}$$

This formula is valid when $x^- < 0 < k^+ < x^+$, the other cases not being of interest to us.

To investigate the asymptotic behavior we will assume $x^- = -x^+$, and vary the value of $b = x^+$ to define the barrier. Figure 1 shows $u(x,t)$ as a function of b in the range $0.25 < b < 1.0$. In this example, $\sigma = 0.4$, $r = 0.1$, $K^- = 0.8$, $K^+ = 1.2$, and four values of t. (These graphs are produced equally by (6.29) or (7.2)). It is clear that u_0 tends to zero quite rapidly as $b \to \infty$.

One may use $u_0^{(1)}$ as an approximation of u_0, i.e., taking only the first summand of (6.29). Since this series converges quite rapidly, this can provide an excellent approximation, as is shown is Figure 2.

The relative portion $|u_0|/u$ can be used as an indication of the effect of barriers on option value. When this is sufficiently small as indicated by calculations such as given here, an investor may consider it justified to exchange a European option for a double barrier option with the same payoff.

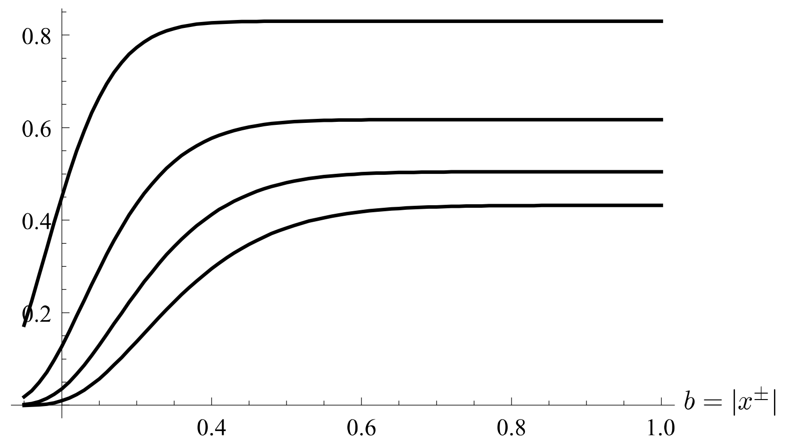

Figure 1. $u(x,t)$ for supershare option evaluated at $x = 0$ for $t = 0.25$ (highest), 0.5, 0.75, 1.0 (lowest). Dotted lines mark the asymptotic value $u_\infty(x,t)$, which does not depend on the barrier.

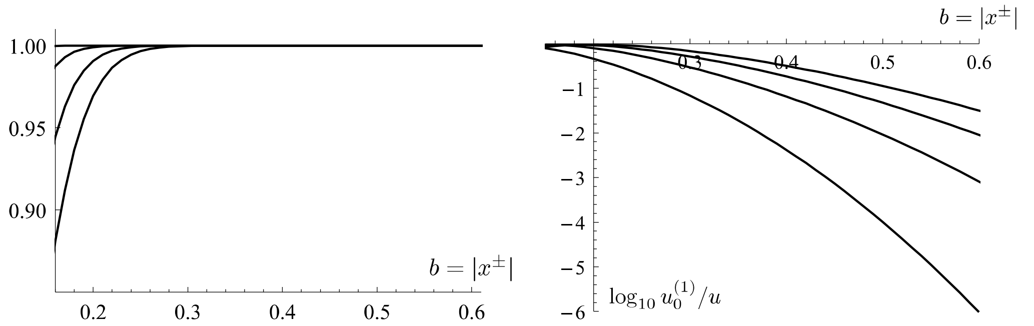

Figure 2. Ratio $u_0^{(1)}(0,t)/u(0,t)$ (left) and order of magnitude (right) for the four times t under consideration.

References

[1] N. I. Akhiezer, *Lectures on Approximation Theory*, Frederick Ungar Publishing Co., New York 1956.

[2] O. E. Barndorff-Neilsen, Processes of normal inverse Gaussian type. *Finance Stoch.* **2**, No. 1 (1998), pp 41-68.

[3] A. Böttcher, S. Grudsky, and R. M. Porter, European double-barrier options with a compound Poisson component. *Progress in Economics Research* **18** (A. Tavidze, ed.), Nova Science Publishers, Huntington, New York (2011), pp 315-331.

[4] M. Boyarchenko and S.I. Boyarchenko, Double barrier options in regime-switching hyper-exponential jump-diffusion models. *Int J. Theor. Appl. Finance* **14**, No. 7 (2011), pp 1005-1043.

[5] M. Boyarchenko and S. Z. Levendorskiĭ, Valuation of continuously monitored double barrier options and related securities. *Math. Finance* **22**, No. 3 (2012), pp 419-444.

[6] M. Boyarchenko and S. Z. Levendorskiĭ, *Non-Gaussian Merton-Black-Scholes Theory*. (Advanced Series on Statistical Science and Applied Probability **9**), *World Scientific Publishing Co., Inc., River Edge,* NJ and Singapore 2002.

[7] M. Boyarchenko and S. Z. Levendorskiĭ, Perpetual American options under Lévy processes. *SIAM Journal Control Optimization* **40**, No. 6 (2002), pp 1663-1696.

[8] M. Boyarchenko and S. Z. Levendorskiĭ, Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability*, **12**, No. 4 (2002), pp 1261-1298.

[9] P. Carr and J. Crosby, A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options. *Quant. Finance* **10**, No. 10 (2010), pp 1115-1136.

[10] E. Eberlein, Application of generalized hyperbolic Lévy motions to finance. *Lévy Processes: Theory and Applications*, O. E. Barndorff-Nielsen, T. Mikosh, and S. Resnil (eds.), Birkhäuser (2001), pp 319-337.

[11] E. Eberlein and J. Jacod, On the range of options prices. *Finance Stoch.* **1** (1997), pp 131-140.

[12] F. Esscher, On the probability function in the collective theory of risk. *Skandinavisk Aktuarietidskrift* **15** (1932), pp 175-95.

[13] M. V. Fedoryuk, *The Saddle Point Method*, Nauka, Moscow (in Russian) 1988.

[14] L. Feng and V. Linetsky, Pricing discretely monitored barrier options and defaultable bonds. *Lévy process models: a fast Hilbert transform approach*, *Math. Finance* **18**, No. 3 (2008), pp 337-384.

[15] H. Geman and M. Yor, Pricing and hedging double-barrier options: a probabilistic approach. *Math. Finance* **6**, No. 4 (1996), pp 365-378.

[16] J. A. Goldstein, *Semigroups of Linear Operators and Applications,* Oxford University Press, New York 1985.

[17] G. A. Grinberg, *A Collection of Problems in the Mathematical Theory of Electrical and Magnetic Phenomena*, Moscow-Leningrad (in Russian) 1948.

[18] C. H. Hui, One-touch double barrier binary option values. *Applied Financial Economics* **6** (1996), pp 343-346.

[19] C. H. Hui, Time dependent barrier option values. *J. Futures Markets* **17** (1997), pp 667-688.

[20] J. D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, New York 1962.

[21] J. Jeans, *The Mathematical Theory of Electricity and Magnetism*, Cambridge University Press, New York 1963.

[22] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin and Heidelberg 1995.

[23] N. E. Kochin, I. A. Kibel', and N. V. Roze, *Theoretical Hydrodynamics*, Interscience (Translated from Russian) 1964.

[24] S. G. Krein, *Linear Differential Equation in Banach Space*, Transl. Math. Monogr. 29 1971.

[25] N. Kunimoto and M. Ikeda, Pricing options with curved boundaries. *Math. Finance* **2** (1997), pp 275-298.

[26] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, New York 1982.

[27] A. Lipton, *Mathematical Methods for Foreign Exchange, A Financial Engineer's Approach*, World Scientific, New Jersy 2001.

[28] D. B. Madan, P. Carr, and E. C. Chang, The variance gamma process and option pricing. *European Finance Review* **2** (1997), pp 79-105.

[29] A. Matacz, Financial modelling and option theory with the truncated Lévy process. *Int. J. Theor. Appl. Finance* **3**, No. 1 (2000), pp 143-160.

[30] E. Mordecki, Optimal stopping and perpetual options for Lévy processes. *Talk presented at the I World Congress of the Bachelier Finance Society* (2000).

[31] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York 1974.

[32] A. Pelsser, Pricing double barrier options using Laplace transform. *Finance Stoch.* **4**, No. 1 (1997), pp 95-104.

[33] M. Potters, J-P. Bouchaud, and L. Laloux, *Theory of Financial Risks: From Statistical Physics to Risk Management*, Cambridge University Press, Cambridge 2000.

[34] R. Raible, *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts. Dissertation*, Mathematische Fakultät, Universität Freiberg im Breisgau 2000.

[35] A. Sepp, Analytical pricing of double-barrier options under a double-exponential jump diffusion process: applications of Laplace transform. *Int. J. Theor. Appl. Finance* **7**, No. 2 (2004), pp 151-175.

[36] A. N. Shiryaev, *Essentials of Stochastic Finance: Facts, Models, Theory*, World Scientific, Singapore-Jersey-London-Hong Kong 1999.

[37] J. Sidenius, Double barrier options: valuation by path counting. *Journal of Computational Finance* **1**, No. 3 (1998), pp 63-79.

[38] P. Wilmott, J. Dewynne, and S. Howison, *Option Pricing: Mathematical Models and Computations*, Oxford Financial Press, Oxford 1993.